On the Role of Canonicity in Knowledge Compilation

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Abstract

Knowledge compilation is a powerful reasoning paradigm with many applications across AI and computer science more broadly. We consider the problem of bottom-up compilation of knowledge bases, which is usually predicated on the existence of a polytime function for combining compilations using Boolean operators (usually called an Apply function). While such a polytime Apply function is known to exist for certain languages (e.g., OBDDs) and not exist for others (e.g., DNNFs), its existence for certain languages remains unknown. Among the latter is the recently introduced language of Sentential Decision Diagrams (SDDs): while a polytime Apply function exists for SDDs, it was unknown whether such a function exists for the important subset of compressed SDDs which are canonical. We resolve this open question in this paper and consider some of its theoretical and practical implications. Some of the findings we report question the common wisdom on the relationship between bottom-up compilation, language canonicity and the complexity of the Apply function.

Introduction

Knowledge compilation is an area of research that has a long tradition in AI; see Cadoli and Donini (1997). Initially, work in this area took the form of searching for tractable languages based on CNFs (e.g. Selman and Kautz; del Val; Marquis (1991; 1994; 1995)). However, the area took a different turn a decade ago with the publication of the “Knowledge Compilation Map” (Darwiche and Marquis 2002). Since then, the work on knowledge compilation became structured across three major dimensions; see Darwiche (2014) for a recent survey: (1) identifying new tractable languages and placing them on the map by characterizing their succinctness and the polytime operations they support; (2) building compilers that map propositional knowledge bases into tractable languages; and (3) using these languages in various applications, such as diagnosis (Elliott and Williams 2006; Huang and Darwiche 2005; Barrett 2005; Siddiqi and Huang 2007), planning (Palacios et al. 2005; Huang 2006), probabilistic reasoning (Chavira, Darwiche, and Jaeger 2006; Chavira and Darwiche 2008; Fierens et al. 2011), and statistical relational learning (Fierens et al. 2013). More recently, knowledge compilation has greatly influenced the area of probabilistic databases (Suciu et al. 2011; Jha and Suciu 2011; Bekkerman, Deshpande, and Getoor 2012; Beame et al. 2013) and became also increasingly influential in first-order probabilistic inference (Van den Broeck et al. 2011; Van den Broeck 2011; Van den Broeck 2013). Another area of influence is in the learning of tractable probabilistic models (Lowd and Roosdenas 2013; Gens and Domingos 2013; Kisa et al. 2014a), as knowledge compilation has formed the basis of a number of recent approaches in this area of research (ICML hosted the First International Workshop on Learning Tractable Probabilistic Models (LTPM) in 2014).

One of the more recent introductions to the knowledge compilation map is the Sentential Decision Diagram (SDD) (Darwiche 2011). The SDD is a target language for knowledge compilation. That is, once a propositional knowledge base is compiled into an SDD, the SDD can be reused to answer multiple hard queries efficiently (e.g., clausal entailment or model counting). SDDs subsume Ordered Binary Decision Diagrams (OBDDs) (Bryant 1986) and come with tighter size bounds (Darwiche 2011; Razgon 2013; Oztok and Darwiche 2014), while still being equally powerful as far as their polytime support for classical queries (e.g., the ones in Darwiche and Marquis (2002)). Moreover, SDDs are a specialization of d-DNNFs (Darwiche 2001), which received much attention over the last decade. Even though SDDs are less succinct than d-DNNFs, they can be compiled bottom-up, just like OBDDs. For example, a clause can be compiled by disjoining the SDDs corresponding to its literals, and a CNF can be compiled by conjoning the SDDs corresponding to its clauses. This bottom-up compilation is implemented using the Apply function, which combines two SDDs using Boolean operators. Bottom-up compilation makes SDDs attractive for several AI applications, in particular for reasoning in probabilistic graphical models (Choi, Kisa, and Darwiche 2013) and probabilistic programs, both exact (Vlasselaer et al. 2014) and approximate (Renkens et al. 2014), as well as tractable learning (Kisa et al. 2014a; 2014b). Bottom-up compilation can be critical when the knowledge base to be compiled is constructed incrementally.

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1Apply originated in the OBDD literature (Bryant 1986).
An Open Problem and its Implications

According to common wisdom, a language supports bottom-up compilation only if it supports a polytime Apply function. For example, OBDDs are known to support bottom-up compilation and have traditionally been compiled this way. In fact, the discovery of SDDs was mostly driven by the need for bottom-up compilation, which was preceded by the discovery of structured decomposability (Pipatsrisawat and Darwiche 2008): a property that enables some Boolean operations to be applied in polynomial time. SDDs satisfy this property and stronger ones, leading to a polytime Apply function (Darwiche 2011). It was unknown, however, whether this function existed for the important subset of compressed SDDs which are canonical. This has been an open question since SDDs were first introduced in (Darwiche 2011).

We resolve this open question in this paper, showing that such an Apply function does not exist in general. We also pursue some theoretical and practical implications of this result, on bottom-up compilation in particular. On the practical side, we reveal an empirical finding that seems quite surprising: bottom-up compilation with compressed SDDs is much more feasible practically than with uncompressed ones, even though the latter supports a polytime Apply function while the former does not. This finding questions common convictions on the relative importance of a polytime Apply in contrast to canonicity as desirable properties for a language that supports efficient bottom-up compilation. On the theoretical side, we show that some transformations (e.g., conditioning) can blow up the size of compressed SDDs, while they do not for uncompressed SDDs.

Technical Background

We will use the following notation for propositional logic. Upper-case letters (e.g., $X$) denote propositional variables and bold letters represent sets of variables (e.g., $X$). A literal is a variable or its negation. A Boolean function $f(X)$ maps each instantiation $x$ of variables $X$ into $\top$ (true) or $\bot$ (false).

The SDD Representation

The SDD can be thought of as a “data structure” for representing Boolean functions since SDDs can be canonical and support a number of efficient operations for constructing and manipulating Boolean functions (Darwiche 2011; Xue, Choi, and Darwiche 2012; Choi and Darwiche 2013).

Partitions

SDDs are based on a new type of Boolean function decomposition, called partitions. Consider a Boolean function $f$ and suppose that we split its variables into two disjoint sets, $X$ and $Y$. We can always decompose the function $f$ as

$$f = \bigvee_{i} p_i(X) \land s_i(Y),$$

where we require that the sub-functions $p_i(X)$ are mutually exclusive, exhaustive, and consistent (non-false). This kind of decomposition is called an $(X, Y)$-partition, and it always exists. The sub-functions $p_i(X)$ are called primes and the sub-functions $s_i(Y)$ are called subs (Darwiche 2011). For an example, consider the function: $f = (A \land B) \lor (B \land C) \lor (C \land D)$. By splitting the function variables into $X = \{A, B\}$ and $Y = \{C, D\}$, we get the following decomposition:

$$f = \left[ p_1(X) \land s_1(Y) \right] \lor \cdots \lor \left[ p_n(X) \land s_n(Y) \right]. \quad (1)$$

The primes are mutually exclusive, exhaustive and non-false. This decomposition is represented by a decision SDD node, which is depicted by a circle $\bigcirc$ as in Figure 1. The above decomposition corresponds to the root decision node in this figure. The children of a decision SDD node are depicted by paired boxes $[\bigcirc \boxed{\quad} \bigcirc \boxed{\quad}]$ called elements. The left box of an element corresponds to a prime $p$, while the right box corresponds to its sub $s$. In the graphical depiction of SDDs, a prime $p$ or sub $s$ are either a constant, literal or pointer to a decision SDD node. Constants and literals are called terminal SDD nodes.

Compression

An $(X, Y)$-partition is compressed when its subs $s_i(Y)$ are distinct. Without the compression property, a function can have many different $(X, Y)$-partitions. However, for a function $f$ and a particular split of the function variables into $X$ and $Y$, there exists a unique compressed $(X, Y)$-partition of function $f$. The $(A, B, C, D)$-partition in (1) is compressed. Its function has another $(A, B, C, D)$-partition, which is not compressed:

$$\{(A \land B, \top), (\neg A \land B, C), (A \land \neg B, D \land C), (\neg A \land \neg B, D \land C)\}. \quad (2)$$

An uncompressed $(X, Y)$-partition can be compressed by merging all elements $(p_1, s), \ldots, (p_n, s)$ that share the same sub into one element $(p_1 \lor \cdots \lor p_n, s)$. Compressing (2) combines the two last elements into $[(A \land \neg B) \lor (\neg A \land \neg B), D \land C]$ and $(\neg B, D \land C)$, resulting in (1). This is the unique compressed $(A, B, C, D)$-partition of $f$. A compressed SDD is one which contains only compressed partitions.

Vtree

An SDD can be defined using a sequence of recursive $(X, Y)$-partitions. To build an SDD, we need to determine which $X$ and $Y$ are used in every partition in the SDD. This process is governed by a vtree: a full, binary tree, whose leaves are labeled with the function variables; see Figures 1b and 2. The root $v$ of the vtree partitions variables into those
Figure 2: Different vtrees over the variables $A$, $B$, $C$, and $D$. The vtree on the left is right-linear.

appearing in the left subtree ($X$) and those appearing in the right subtree ($Y$). This implies an ($X, Y$)-partition $\beta$ of the Boolean function, leading to the root SDD node (we say in this case that partition $\beta$ is normalized for vtree node $v$). The primes and subs of this partition are turned into SDDs, recursively, using vtree nodes from the left and right subtrees. The process continues until we reach variables or constants (i.e., terminal SDD nodes). The vtree used to construct an SDD can have a dramatic impact on the SDD, sometimes leading to an exponential difference in the SDD size.

**Two Forms of Canonicity** Even though compressed ($X, Y$)-partitions are unique for a fixed $X$ and $Y$, we need one of two additional properties for a compressed SDD to be unique (i.e., canonical) given a vtree:

- **Normalization**: If an ($X, Y$)-partition $\beta$ is normalized for vtree node $v$, then the primes (subs) of $\beta$ must be normalized for the left (right) child of $v$—as opposed to a left (right) descendant of $v$.

- **Trimming**: The SDD contains no ($X, Y$)-partitions of the form $\{(\top, \alpha)\}$ or $\{(\alpha, \top), (\neg \alpha, \bot)\}$.

For a Boolean function, and a fixed vtree, there is a unique compressed, normalized SDD. There is also a unique compressed, trimmed SDD (Darwiche 2011). Thus, both representations are canonical, although trimmed SDDs tend to be smaller. One can trim an SDD by replacing ($X, Y$)-partitions of the form $\{(\top, \alpha)\}$ or $\{(\alpha, \top), (\neg \alpha, \bot)\}$ with $\alpha$. One can normalize an SDD by adding intermediate partitions of the same form. Since these translations are efficient, our theoretical results will apply to both canonical representations. In what follows, we will restrict our attention to compressed, trimmed SDDs and refer to them as **canonical SDDs**.

**SDDs and OBDDs** OBDDs correspond precisely to SDDs that are constructed using a special type of vtree, called a right-linear vtree (Darwiche 2011); see Figure 2. The left child of each inner node in these vtrees is a variable. With right-linear vtrees, compressed, trimmed SDDs correspond to reduced OBDDs, while compressed, normalized SDDs correspond to oblivious OBDDs (Xue, Choi, and Darwiche 2012) (reduced and oblivious OBDDs are also canonical). The size of an OBDD depends critically on the underlying variable order. Similarly, the size of an SDD depends critically on the vtree used (right-linear vtrees correspond to variable orders). Vtree search algorithms can sometimes find SDDs that are orders-of-magnitude more succinct than OBDDs found by searching for variable orders (Choi and Darwiche 2013). Such algorithms assume canonical SDDs, allowing one to search the space of SDDs by searching the space of vtrees instead.

**Queries** SDDs are a strict subset of deterministic, decomposable negation normal form (d-DNNF). They are actually a strict subset of structured d-DNNF and, hence, support the same polytime queries supported by structured d-DNNF (Pipatsrisawat and Darwiche 2008); see Table 1. We defer the reader to Darwiche and Marquis (2002) for a detailed description of the queries typically considered in knowledge compilation. This makes SDDs as powerful as OBDDs in terms of their support for certain queries (e.g., clausal entailment, model counting, and equivalence checking).

**Bottom-up Construction** SDDs are typically constructed in a bottom-up fashion. For example, to construct an SDD for the function $f = (A \land B) \lor (B \land C) \lor (C \land D)$, we first retrieve terminal SDDs for the literals $A$, $B$, $C$, and $D$. We then **conjoin** the terminal SDD for literal $A$ with the one for literal $B$, to obtain an SDD for the term $A \land B$. The process is repeated to obtain SDDs for the terms $B \land C$ and $C \land D$. The resulting SDDs are then **disjoined** to obtain an SDD for the whole function. These operations are not all efficient on structured d-DNNFs. However, SDDs satisfy stronger properties than structured d-DNNFs, allowing one, for example, to conjoin or disjoin two SDDs in polytime.

This bottom-up compilation is performed using the **Apply** function. Algorithm 1 outlines an **Apply** function that takes two SDDs $\alpha$ and $\beta$, and a binary Boolean operator $\circ$ (e.g., $\land$, $\lor$, xor), and returns the SDD for $\alpha \circ \beta$ (Darwiche 2011). Line 13 optionally compresses each partition, in order to return a compressed SDD. Without compression, this algorithm has a time and space complexity of $O(nm)$, where $n$ and $m$ are the sizes of input SDDs. This comes at the expense of losing canonicity. Whether a polytime complexity can be attained under compression was an open question.

There are several implications of this question. For example, depending on the answer, one would know whether certain transformations, such as conditioning and existential

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**Table 1: Analysis of supported queries, following Darwiche and Marquis (2002).** $\checkmark$ means that a polytime algorithm exists for the corresponding language/query, while $\circ$ means that no such algorithm exists unless $P = NP$. Where applicable, depending on the answer, one would know whether certain transformations, such as conditioning and existential.

<table>
<thead>
<tr>
<th>Query</th>
<th>Description</th>
<th>OBDD</th>
<th>SDD</th>
<th>d-DNNF</th>
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<td>$\checkmark$</td>
<td>$\checkmark$</td>
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<td>model enumeration</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
</tr>
</tbody>
</table>

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2This code assumes that the SDD is normalized. The **Apply** for trimmed SDDs is similar, although a bit more technically involved.
There exists a class of Boolean functions.

**Theorem 1.** There exists a class of Boolean functions $f_m(X_1, \ldots, X_m)$ and corresponding vtrees $T_m$ such that $f_m$ has an SDD of size $O(m^2)$ wrt vtree $T_m$, yet the canonical SDD of function $f_m$ wrt vtree $T_m$ has size $\Omega(2^m)$.

The proof is constructive, identifying a class of functions $f_m$ with the given properties. The functions $f^*_m(X, Y, Z) = \bigvee_{i=1}^m (\bigwedge_{j=1}^{i-1} \neg Y_j) \wedge Y_i \wedge X_i$ have $2m+1$ variables. Of these, $Z$ is non-essential. Consider a vtree $T_m$ of the form

```
  1
 / \
2   Z
/ \   /
X  Y
```

where the sub-vtrees over variables $X$ and $Y$ are arbitrary.

We now construct an uncompressed SDD for this function using vtree $T_m$ and whose size is $O(m^2)$. We will then show that the compressed SDD for this function and vtree has a size $\Omega(2^m)$.

The first step is to construct a partition of function $f^*_m$ that respects the root vtree node, that is, an $(XY,Z)$-partition.

**Algorithm 1** $\text{Apply}(\alpha, \beta, \circ)$

1: if $\alpha$ and $\beta$ are constants or literals then
2: return $\alpha \circ \beta$  // result is a constant or literal
3: else if $\text{Cache}(\alpha, \beta, \circ) \neq \text{nil}$ then
4: return $\text{Cache}(\alpha, \beta, \circ)$  // has been computed before
5: else
6: $\gamma \leftarrow \{\}$
7: for all elements $(p_i, s_i)$ in $\alpha$ do
8:  for all elements $(q_j, r_j)$ in $\beta$ do
9:    $p \leftarrow \text{Apply}(p_i, q_j, \wedge)$
10:       if $p$ is consistent then
11:          $s \leftarrow \text{Apply}(s_i, r_j, \circ)$
12:          add element $(p, s)$ to $\gamma$
13:     (optionally) $\gamma \leftarrow \text{Compress}(\gamma)$  // compression
14:      // get unique decision node and return it
15: return $\text{Cache}(\alpha, \beta, \circ) \leftarrow \text{UniqueD}(\gamma)$

Quantification, can be supported in polytime on canonical SDDs. Moreover, according to common wisdom, a negative answer may preclude bottom-up compilation from being feasible on canonical SDDs. We answer this question and explore its implications next.

**Complexity of Apply on Canonical SDDs**

The size of a decision node is the number of its elements, and the size of an SDD is the sum of sizes attained by its decision nodes. We now show that compression, given a fixed vtree, may blow up the size of an SDD.

**Theorem 1.** There exists a class of Boolean functions $f_m(X_1, \ldots, X_m)$ and corresponding vtrees $T_m$ such that $f_m$ has an SDD of size $O(m^2)$ wrt vtree $T_m$, yet the canonical SDD of function $f_m$ wrt vtree $T_m$ has size $\Omega(2^m)$.

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/ \   /
X  Y
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where the sub-vtrees over variables $X$ and $Y$ are arbitrary.

We now construct an uncompressed SDD for this function using vtree $T_m$ and whose size is $O(m^2)$. We will then show that the compressed SDD for this function and vtree has a size $\Omega(2^m)$.

The first step is to construct a partition of function $f^*_m$ that respects the root vtree node, that is, an $(XY,Z)$-partition.

Consider

$$\left\{ (Y_1 \wedge X_1, \top), (-Y_1 \wedge Y_2 \wedge X_2, \top), \ldots, (-Y_1 \wedge \cdots \wedge -Y_{m-1} \wedge Y_m \wedge X_m, \top), (Y_1 \wedge -X_1, \bot), (-Y_1 \wedge Y_2 \wedge -X_2, \bot), \ldots, (-Y_1 \wedge \cdots \wedge -Y_{m-1} \wedge Y_m \wedge -X_m, \bot), (-Y_1 \wedge \cdots \wedge -Y_m, \bot) \right\}$$

which is equivalently written as

$$\bigcup_{i=1}^m \left\{ \bigwedge_{j=1}^{i-1} \neg Y_j \wedge Y_i \wedge X_i, \top \right\} \cup \left\{ \bigwedge_{j=1}^m \neg Y_j, \bot \right\}$$

The size of this partition is $2m+1$, and hence linear in $m$. It is *uncompressed*, because there are $m$ elements that share sub $\top$ and $m+1$ elements that share sub $\bot$. The sub already respect the leaf vtree node labeled with variable $Z$.

In a second step, each prime above is written as a compressed $(X,Y)$-partition that respects the left child of the vtree root. Prime $\bigwedge_{j=1}^{i-1} \neg Y_j \wedge Y_i \wedge X_i$ becomes

$$\left\{ \left( X_i, \bigwedge_{j=1}^{i-1} \neg Y_j \wedge Y_i \right), (-X_i, \bot) \right\},$$

prime $\bigwedge_{j=1}^{i-1} \neg Y_j \wedge Y_i \wedge -X_i$ becomes

$$\left\{ \left( -X_i, \bigwedge_{j=1}^{i-1} \neg Y_j \wedge Y_i \right), (X_i, \bot) \right\}$$

and prime $\bigwedge_{j=1}^m \neg Y_j$ becomes

$$\left\{ (\top, \bigwedge_{j=1}^m \neg Y_j) \right\}.$$
first observe that the unique, compressed \((XY,Z)\)-partition of function \(f^m_{m}\) is
\[
\left\{ \begin{array}{l}
  \left( \bigvee_{i=1}^{m} \left( \bigwedge_{j=1}^{i-1} \neg Y_j \right) \wedge Y_i \wedge X_i, \top \right), \\
  \left( \left[ \bigvee_{i=1}^{m} \left( \bigwedge_{j=1}^{i-1} \neg Y_j \right) \wedge Y_i \wedge \neg X_i \right] \vee \left[ \bigwedge_{j=1}^{m} \neg Y_j \right], \bot \right) \end{array} \right. 
\]
Its first prime is the function
\[
f^b_{m}(X, Y) = \bigvee_{i=1}^{m} \left( \bigwedge_{j=1}^{i-1} \neg Y_j \right) \wedge Y_i \wedge X_i,
\]
which we need to represent as an \((X, Y)\)-partition to respect left child of the vtree root. However, Xue, Choi, and Darwiche (2012) proved the following.

**Lemma 2.** The compressed \((X, Y)\)-partition of \(f^b_{m}(X, Y)\) has \(2^m\) elements.

This becomes clear when looking at the function \(f^b_{m}\) after instantiating the \(X\)-variables. Each distinct \(x\) results in a unique subfunction \(f^b_{m}(x, Y)\), and all states \(x\) are mutually exclusive and exhaustive. Therefore,
\[
\{ (x, f^b_{m}(x, Y)) \mid x \text{ instantiates } X \}
\]
is the unique, compressed \((X, Y)\)-partition of function \(f^b_{m}(X, Y)\), and it has \(2^m\) elements. Hence, the compressed SDD must have size \(\Omega(2^m)\).

Theorem 1 has a number of implications, which are summarized in Table 2; see also Darwiche and Marquis (2002).

**Theorem 3.** The results in Table 2 hold.

First, combining two canonical SDDs (e.g., using the conjoin or disjoin operator) may lead to a canonical SDD whose size is exponential in the size of inputs. Hence, if we activate compression in Algorithm 1, the algorithm may take exponential time in the worst-case. Second, conditioning a canonical SDD on a literal may exponentially increase its size (assuming the result is also canonical). Third, forgetting a variable (i.e., existentially quantifying it) from a canonical SDD may exponentially increase its size (again, assuming that the result is also canonical). The proof of this theorem is in the full version of this paper.

Note that these theorems consider the same vtree for both the compressed and uncompressed SDD. They do not pertain to the complexity of compression and \textbf{Apply} when the vtree is allowed to change. In practice, dynamic vtree search is performed in between conditioning and \textbf{Apply}, but not during (vtree search itself calls \textbf{Apply}). Therefore, the setting where the vtree does not change is more accurate to describe the practical complexity of these operations.

These results may seem discouraging. However, we argue next that, in practice, working with canonical SDDs is actually favorable despite the lack of polytime guarantees on these transformations.

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<table>
<thead>
<tr>
<th>Notation</th>
<th>Transformation</th>
<th>SDD</th>
<th>Canonical SDD</th>
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<td>CD</td>
<td>conditioning</td>
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<td>●</td>
</tr>
<tr>
<td>FO</td>
<td>forgetting</td>
<td>●</td>
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<td>singleton forgetting</td>
<td>✓</td>
<td>●</td>
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<td>●</td>
</tr>
<tr>
<td>¬C</td>
<td>negation</td>
<td>●</td>
<td>●</td>
</tr>
</tbody>
</table>

Table 2: Analysis of supported transformations, following Darwiche and Marquis (2002). ✓ means "satisfies"; ● means "does not satisfy". Satisfaction means the existence of a polytime algorithm that implements the transformation.

Our proof of Theorem 1 critically depends on the ability of a vtree to split the variables into arbitrary sets \(X\) and \(Y\). In the full paper, we define a class of \textit{bounded vtrees} where such splits are not possible. Moreover, we show that the subset of SDDs for such vtrees do support polytime \textbf{Apply} even under compression. Right-linear vtrees, which induce an OBDD, are a special case.

**Canonicity or a Polytime \textbf{Apply}?**

One has two options when working with SDDs. The first option is to work with uncompressed SDDs, which are not canonical, but are supported by a polytime \textbf{Apply} function. The second option is to work with compressed SDDs, which are canonical but lose the advantage of a polytime \textbf{Apply} function. The classical reason for seeking canonicity is that it leads to a very efficient equivalence test, which takes constant time (both compressed and uncompressed SDDs support a polytime equivalence test, but the one known for uncompressed SDDs is not a constant time test). The classical reason for seeking a polytime \textbf{Apply} function is to enable bottom-up compilation, that is, compiling a knowledge base (e.g., CNF or DNF) into an SDD by repeated application of the \textbf{Apply} function to components of the knowledge base (e.g., clauses or terms). If our goal is efficient bottom-up compilation, one may expect that uncompressed SDDs provide a better alternative. However, our next empirical results suggest otherwise. Our goal in this section is to shed some light on this phenomena through some empirical evidence and then an explanation.

We used the SDD package provided by the Automated Reasoning Group at UCLA in our experiments. The package works with compressed SDDs, but can be adjusted to work with uncompressed SDDs as long as dynamic vtree search is not invoked. In our first experiment, we compiled CNFs from the LGSynth89 benchmarks into the following (all trimmed):

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3Available at http://reasoning.cs.ucla.edu/
Compressed SDDs respecting an arbitrary vtree. Dynamic vtree search is used to minimize the size of the SDD during compilation, starting from a balanced vtree.

- Compressed SDDs respecting a fixed balanced vtree.

- Uncompressed SDDs respecting a fixed balanced vtree.

Table 3 shows the corresponding sizes and compilation times. According to these results, uncompressed SDDs end up several orders of magnitude larger than the compressed ones, with or without dynamic vtree search. For the harder problems, this translates to orders-of-magnitude increase in compilation times. Often, we cannot even compile the input without reduction (due to running out of 4GB of memory), even on relatively easy benchmarks. For the easiest benchmarks, dynamic vtree search is slower due to the overhead, but yields smaller compilations. The benefit of vtree search shows only in harder problems (e.g., “unreg”).

Next, we consider the harder set of ISCAS89 benchmarks. Of the 17 ISCAS89 benchmarks that compile with compressed SDDs, only one (s27) could be compiled uncompressed SDDs (others run out of memory). That benchmark has a compressed SDD+s size of 108, a compressed SDD size of 315, and an uncompressed SDD size of 4,551. These experiments clearly show the advantage of compressed SDDs over uncompressed ones, even though the latter supports a polytime `Apply` function while the former does not. This begs an explanation and we provide one next that we back up by additional experimental results.

The benefit of compressed SDDs is canonicity, which plays a critical role in the performance of the `Apply` function. Consider in particular Line 4 of Algorithm 1. The test `Cache(α, β, ϕ) ≠ nil` checks whether SDDs α and β have been previously combined using the Boolean operator ϕ. Without canonicity, it is possible that we would have combined some α′ and β′ using ϕ, where SDD α′ is equivalent to, but distinct from SDD α (and similarly for β′ and β). This redundancy also happens when α is not equivalent to α′ (and similarly for β and β′), α◦β is equivalent to α′◦β′, but the result returned by `Apply(α, β, ϕ)` is distinct from the one returned by `Apply(α′, β′, ϕ)`. Two observations are due here. First, this redundancy is still under control when calling `Apply` only once: `Apply` runs in $O(nm)$ time, where $n$ and $m$ are the sizes of input SDDs. However, this redundancy becomes problematic...
when calling Apply multiple times (as in bottom-up compilation), in which case quadratic performance is no longer as attractive. For example, if we use Apply to combine \(k\) SDDs of size \(n\) each, all we can say is that the output will be of size \(O(n^k)\). The second observation is that the previous redundancy will not occur when working with compressed SDDs due to canonicity: Two SDDs are equivalent iff they are represented by the same structure in memory.\(^7\)

This analysis points to the following conclusion: While Apply has a quadratic complexity on uncompressed SDDs, it may have a worse average complexity than Apply on compressed SDDs. Our next experiment is indeed directed towards this hypothesis.

For all benchmarks in Table 3 that can be compiled without vtree search, we intercept all non-trivial calls to Apply (when \(|\alpha| \cdot |\beta| > 500\)) and report the size of the output \(|\alpha \circ \beta|\) divided by \(|\alpha| \cdot |\beta|\). For uncompressed SDDs, we know that \(|\alpha \circ \beta| = O(|\alpha| \cdot |\beta|)\) and that these ratios are therefore bounded above by some constant. For compressed SDDs, however, Theorem 3 states that there exists no constant bound.

Figure 3 shows the distribution of these ratios for the two methods (note the log scale). The number of function calls is 67,809 for compressed SDDs, vs. 1,626,591 for uncompressed ones. The average ratio is 0.027 for compressed, vs. 0.101 for uncompressed. Contrasting the theoretical bounds, compressed Apply incurs much smaller blowups than uncompressed Apply. This is most clear for ratios in the range 0.48, 0.56, covering 30% of the uncompressed, but only 2% of the compressed calls.

The results are similar when looking at runtime for individual Apply calls, which we measure by the number of recursive Apply calls \(r\). Figure 4 reports these, again relative to \(|\alpha| \cdot |\beta|\). The ratio \(r/(|\alpha| \cdot |\beta|)\) is on average 0.013 for compressed SDDs, vs. 0.034 for uncompressed ones. These results corroborate our earlier analysis, suggesting that canonicity is quite important for the performance of bottom-up compilers as they make repeated calls to the Apply function. In fact, this can be more important than a polytime Apply, perhaps contrary to common wisdom which seems to emphasize the importance of polytime Apply in effective bottom-up compilation (e.g., Pipatsrisawat and Darwiche (2008)).

Conclusions

We have shown that the Apply function on compressed SDDs can take exponential time in the worst case, resolving a question that has been open since SDDs were first introduced. We have also pursued some of the theoretical and practical implications of this result. On the theoretical side, we showed that it implies an exponential complexity for various transformations, such as conditioning and existential quantification. On the practical side, we argued empirically that working with compressed SDDs remains favorable, despite the polytime complexity of the Apply function on uncompressed SDDs. The canonicity of compressed SDDs, we argued, is more valuable for bottom-up compilation than a polytime Apply due to its role in facilitating caching and dynamic vtree search. Our findings appear contrary to some of the common wisdom on the relationship between bottom-up compilation, canonicity and the complexity of the Apply function.

Acknowledgments

We thank Arthur Choi, Doga Kisa, Umut Oztok, and Jessa Bekker for helpful suggestions. This work was supported by ONR grant #N00014-12-1-0423, NSF grants #IIS-1118122 and #IIS-0916161, and the Research Foundation-Flanders (FWO-Vlaanderen). GVdB is also at KU Leuven, Belgium.

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