Sentential Decision Diagrams and their Applications

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Arthur Choi, and Adnan Darwiche

UCLA

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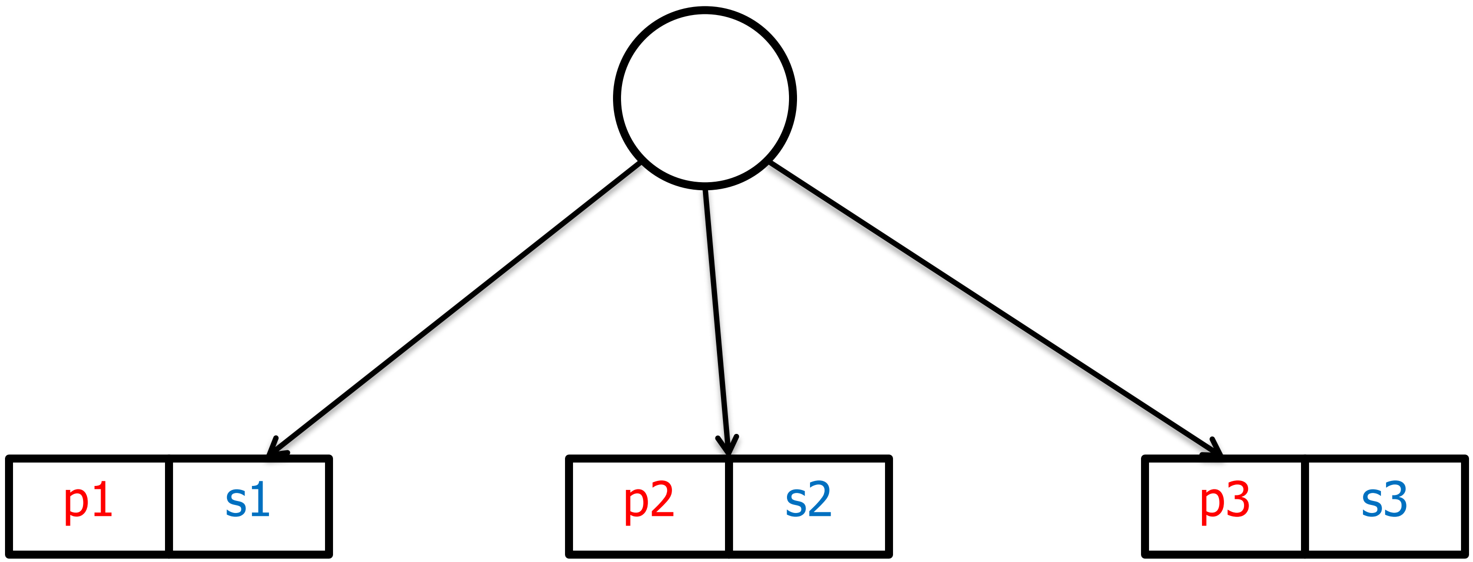
Basing Decisions on Sentences

US Senate: 54 Rep., 44 Dem., and 2 Indep.



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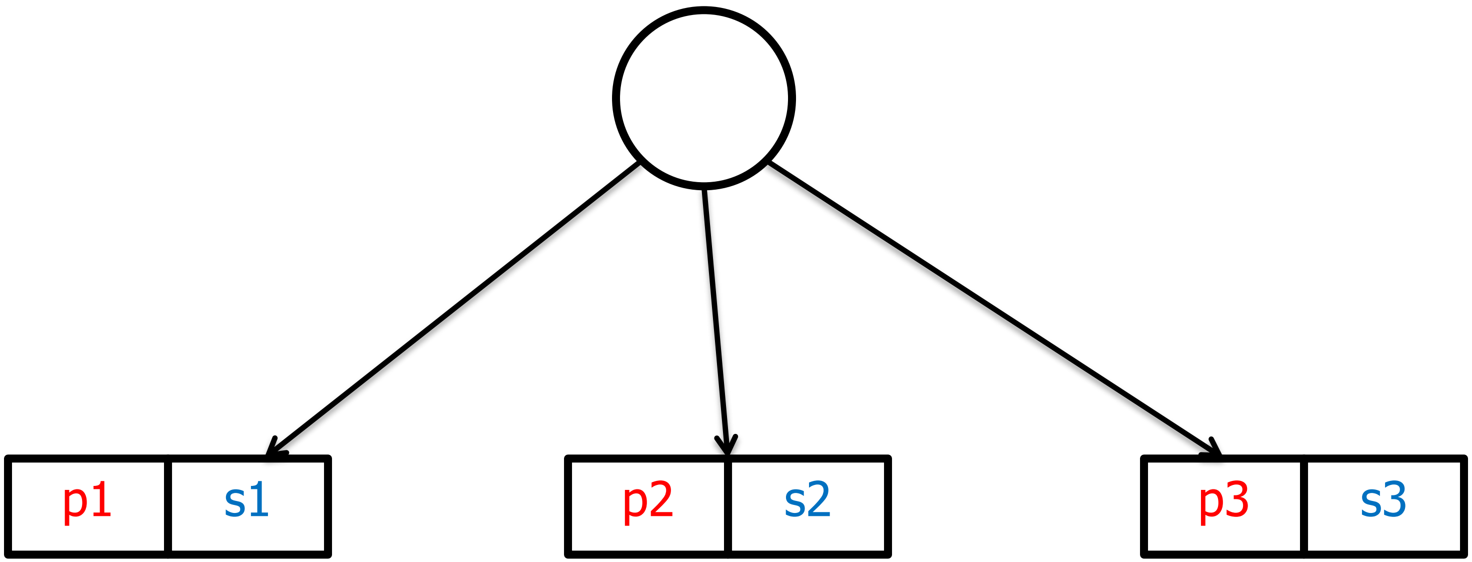

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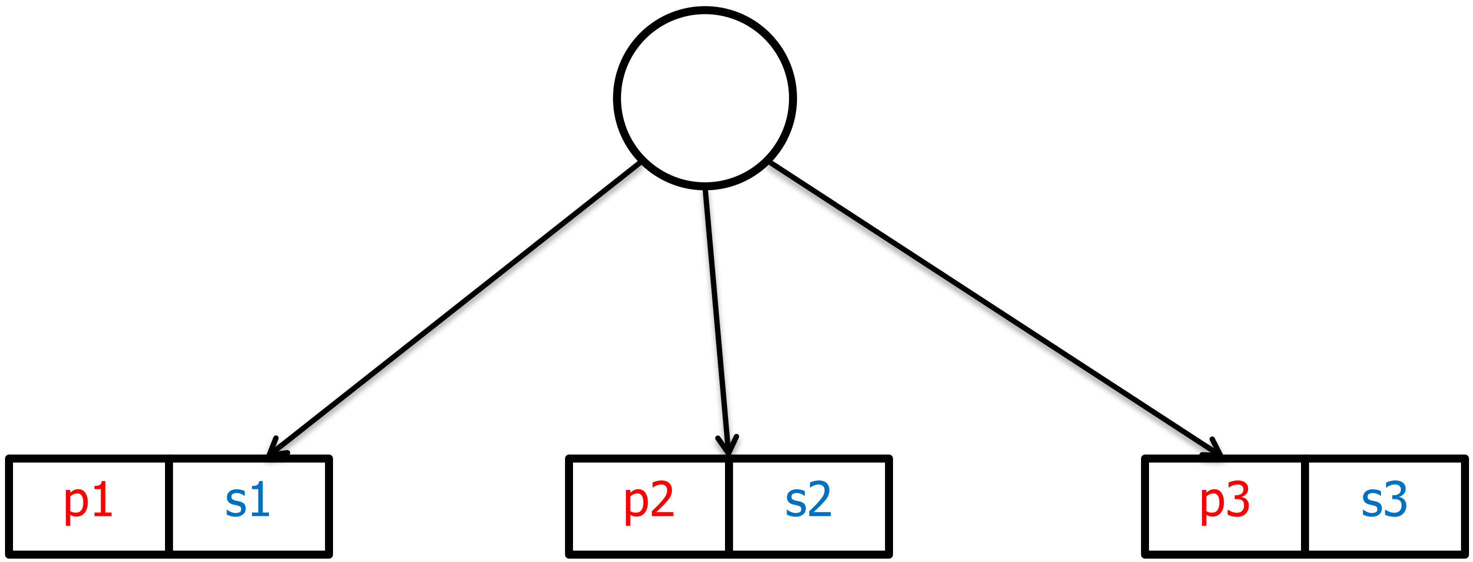

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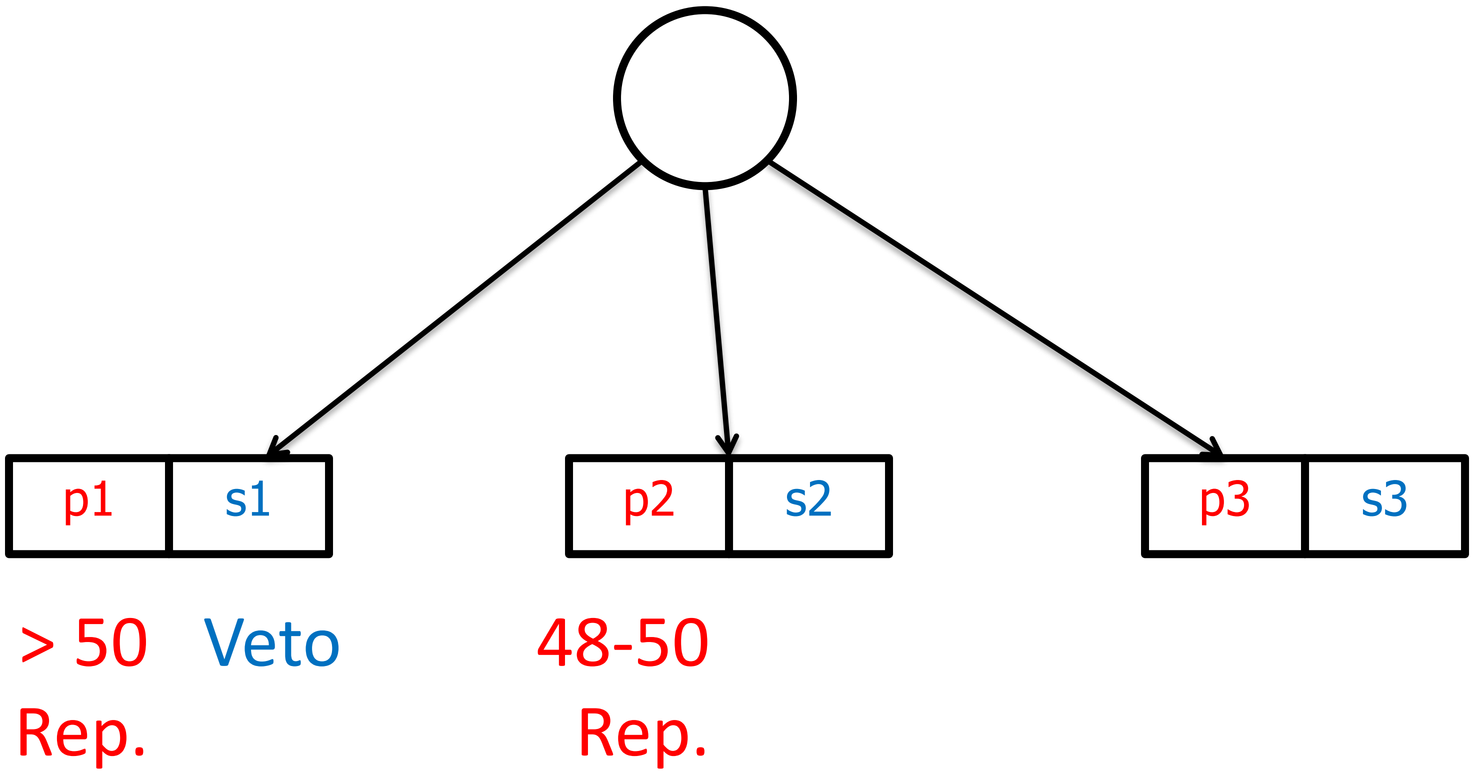

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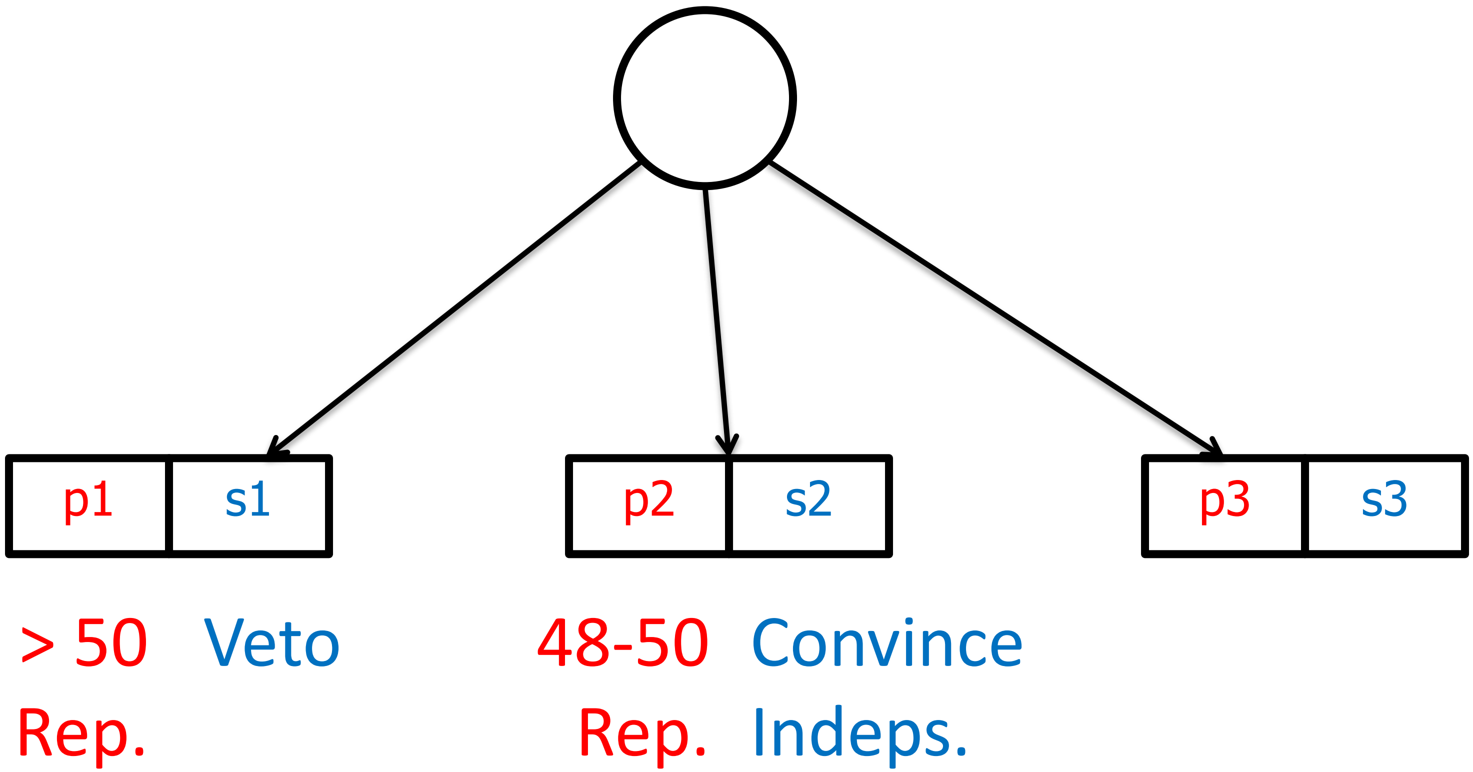

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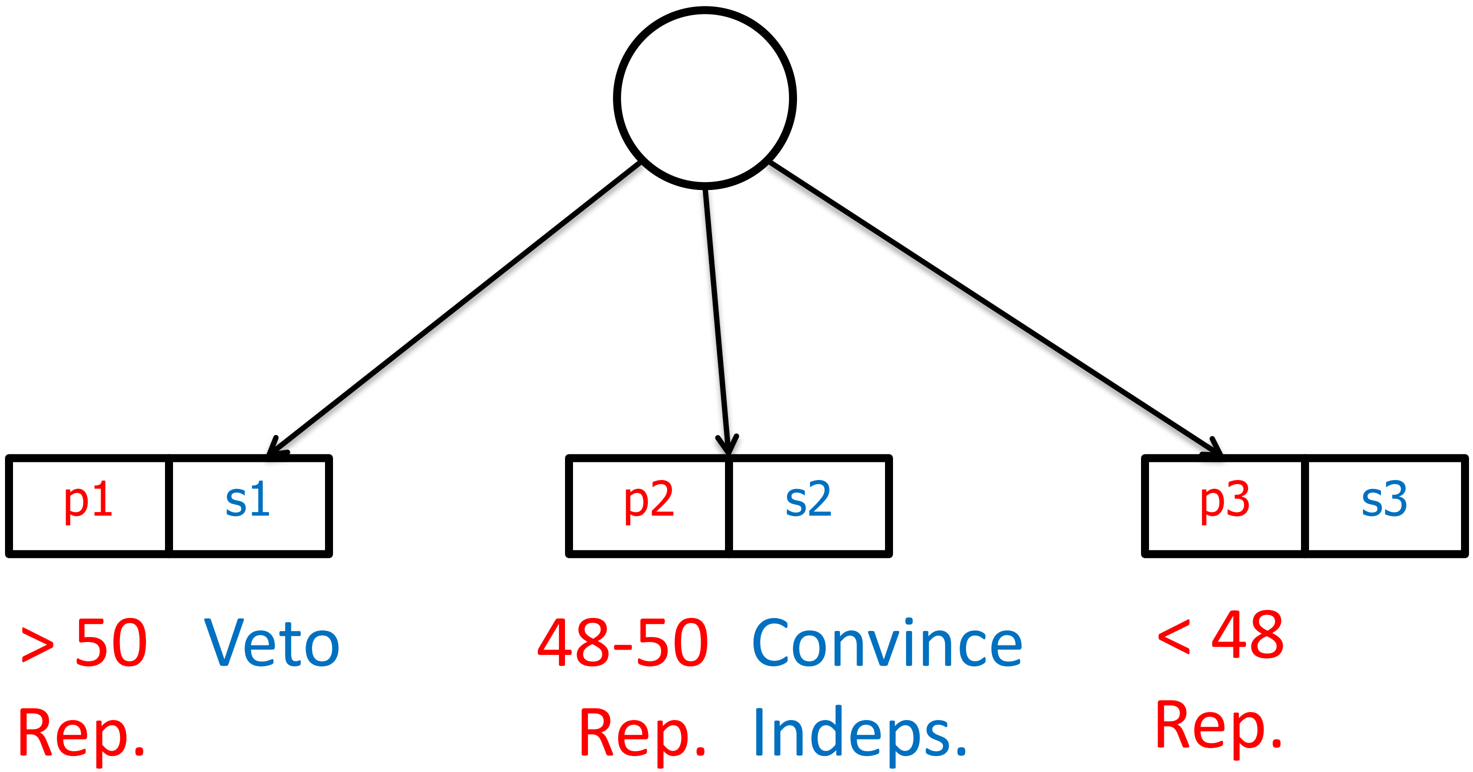

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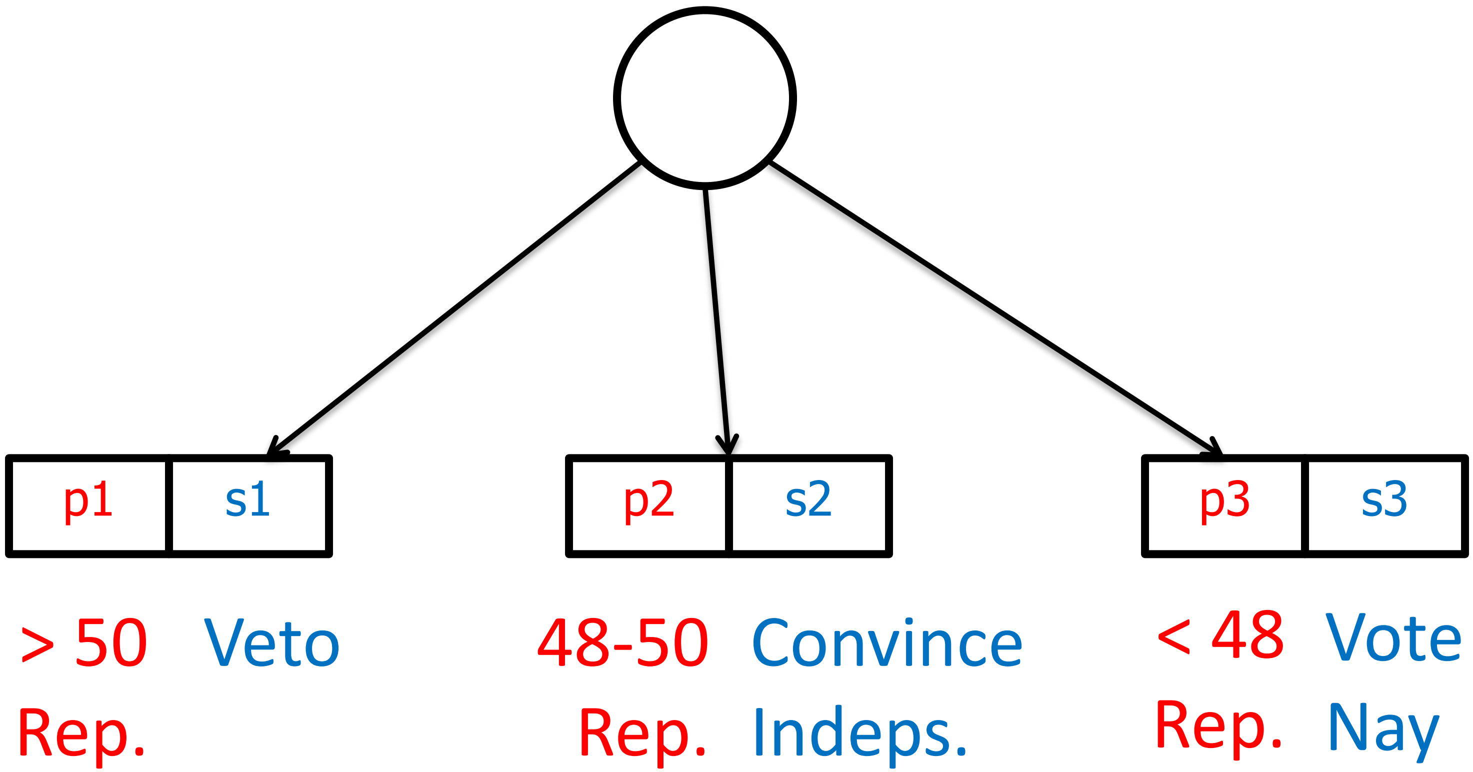

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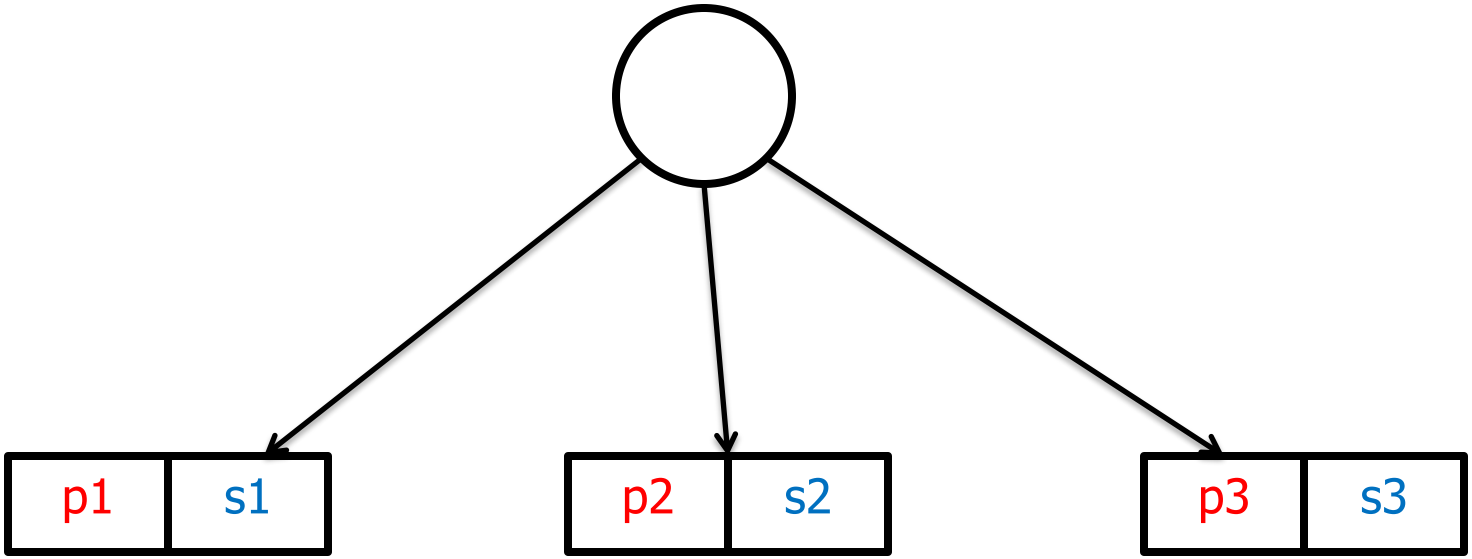

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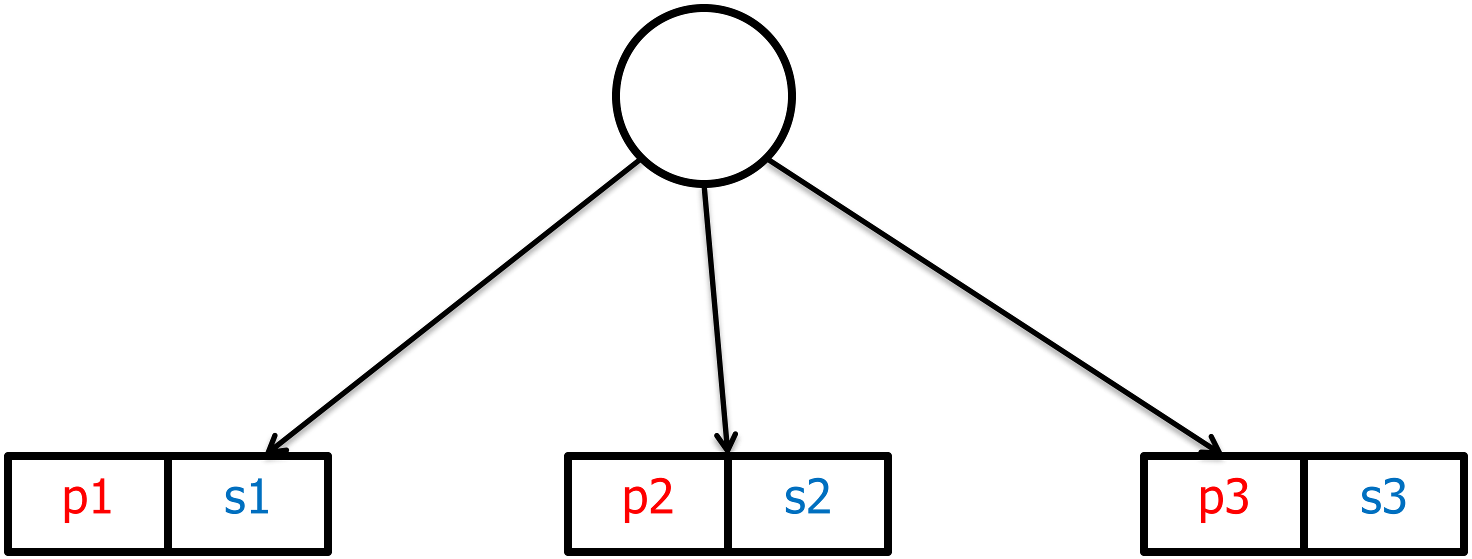
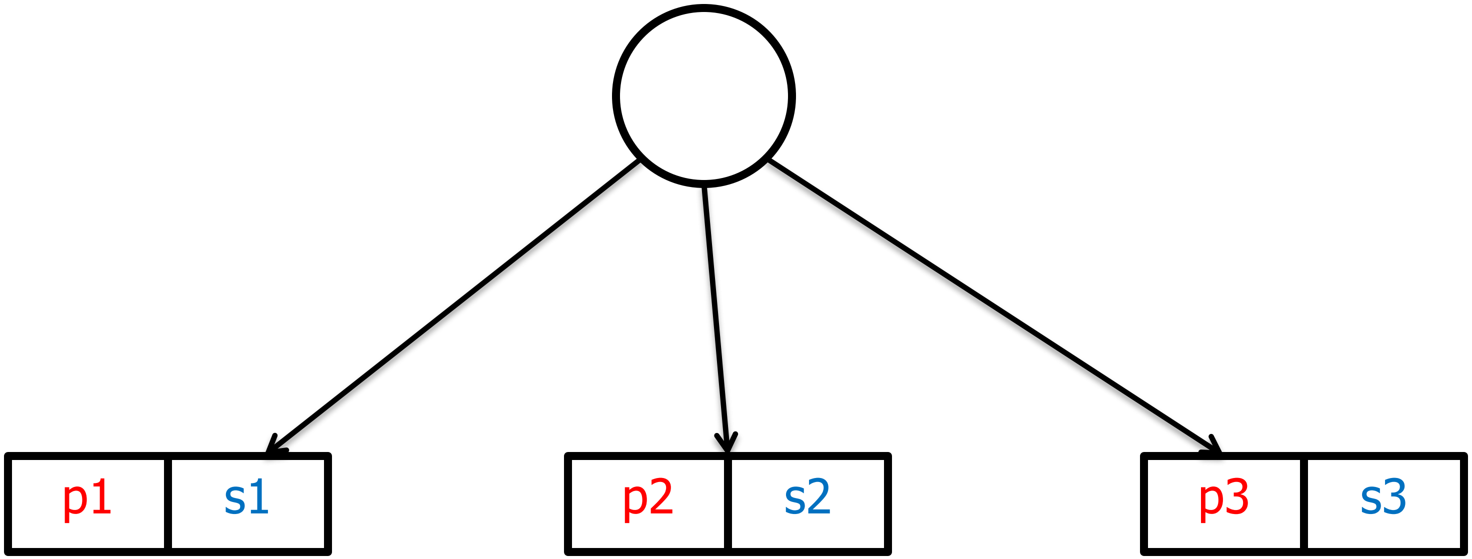


Branch on **sentences** p1, p2, and p3:

Basing Decisions on Sentences



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- p1, p2, p3 are **mutually exclusive**, **exhaustive** and not false

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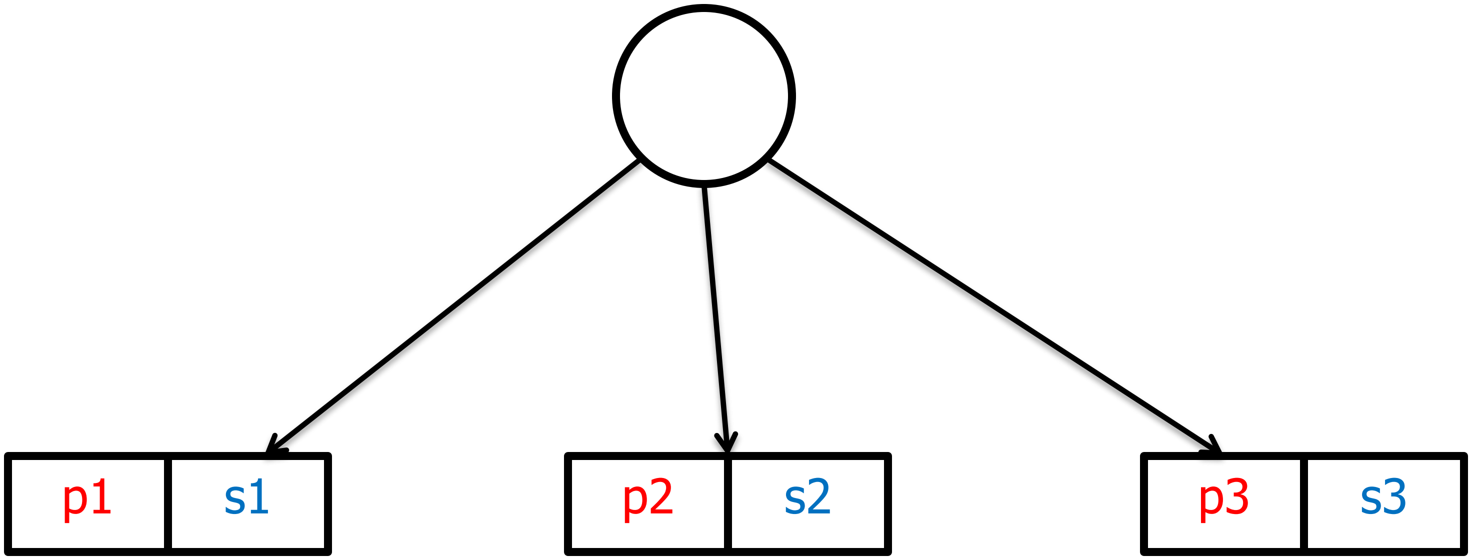

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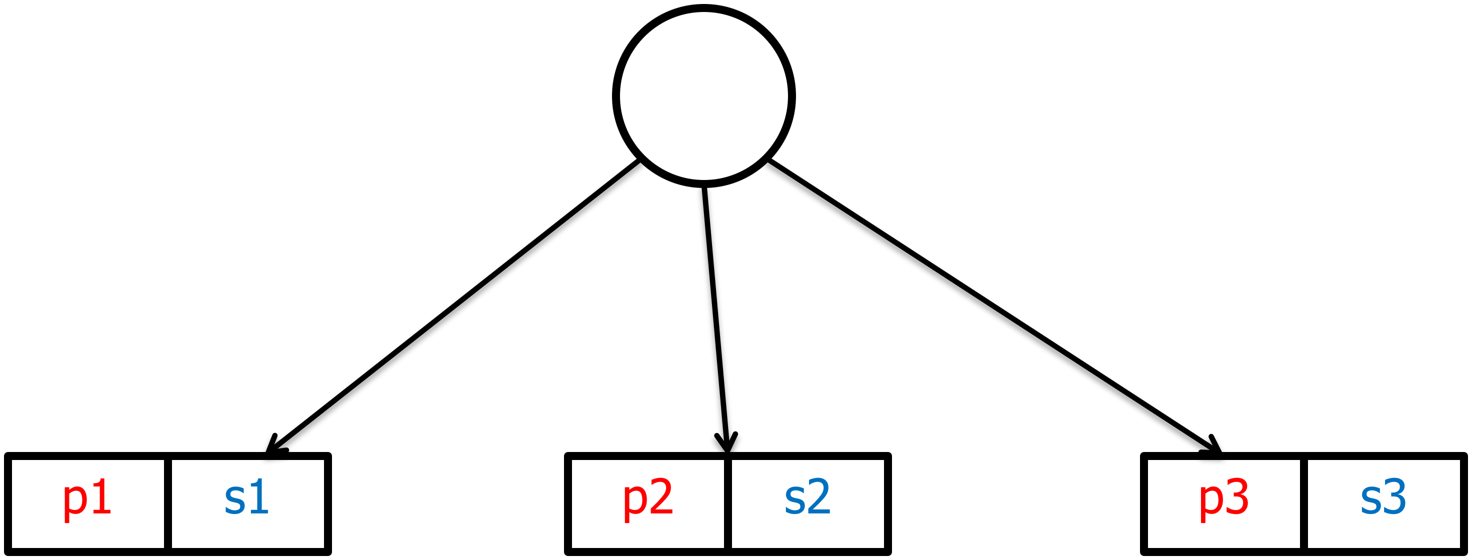
- p1, p2, p3 are **mutually exclusive**, **exhaustive** and not false
- p1, p2, p3 are called **primes** and represented by SDDs

Basing Decisions on Sentences

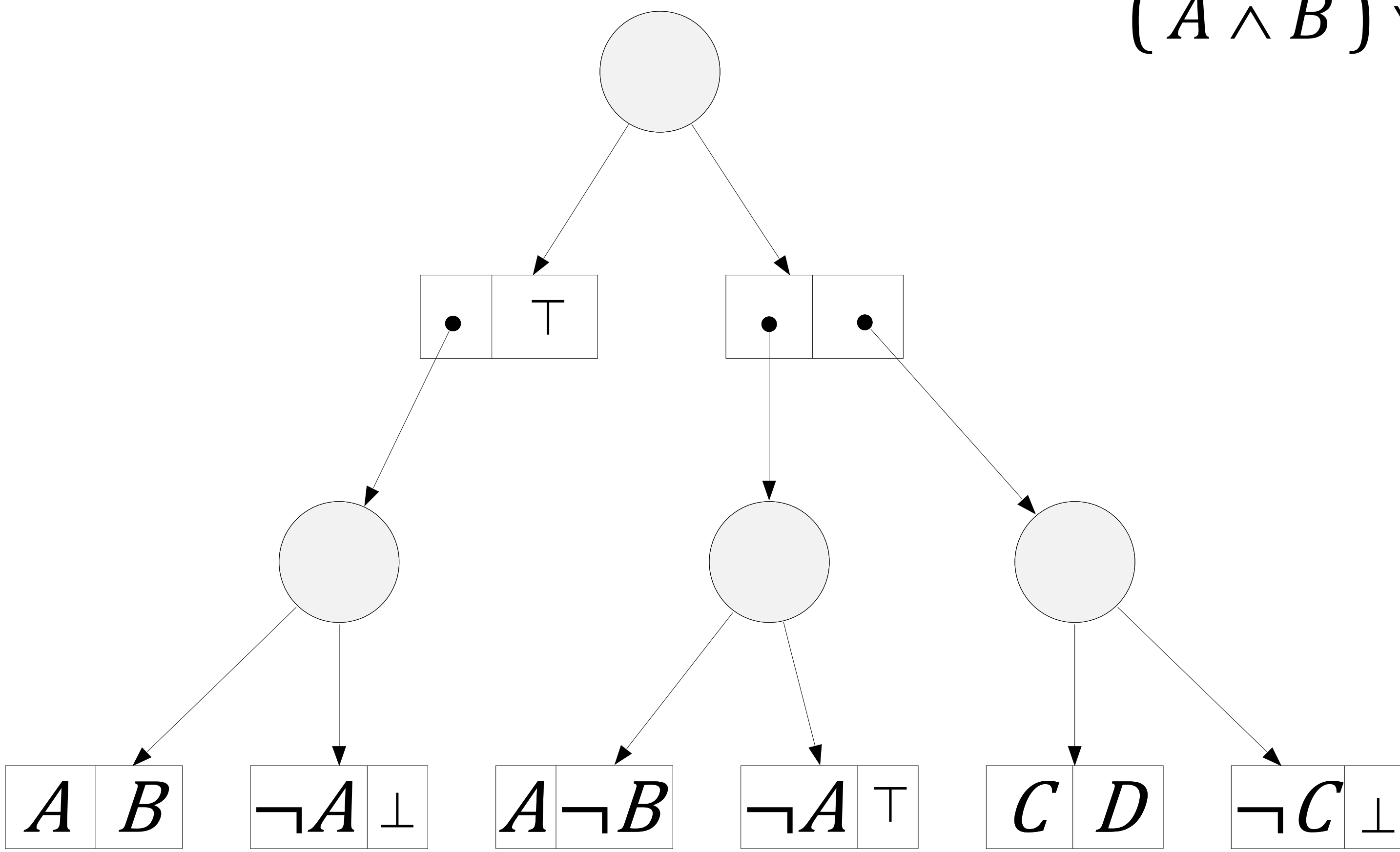


Branch on **sentences** p1, p2, and p3:

- p1, p2, p3 are **mutually exclusive**, **exhaustive** and not false
- p1, p2, p3 are called **primes** and represented by SDDs
- s1, s2, s3 are called **subs** and represented by SDDs

Basing Decisions on Sentences

$$f(A, B, C, D) = (A \wedge B) \vee (C \wedge D)$$



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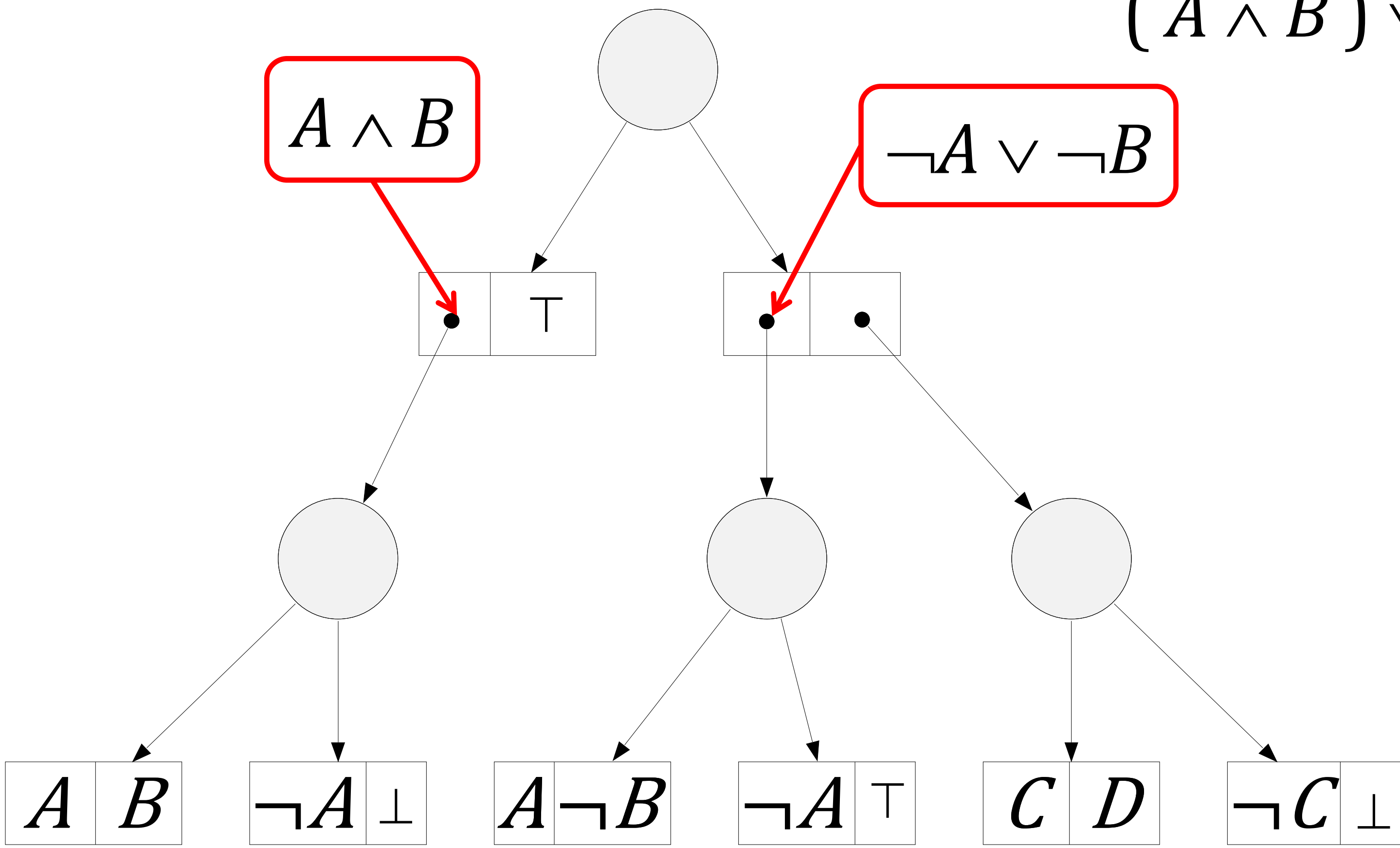

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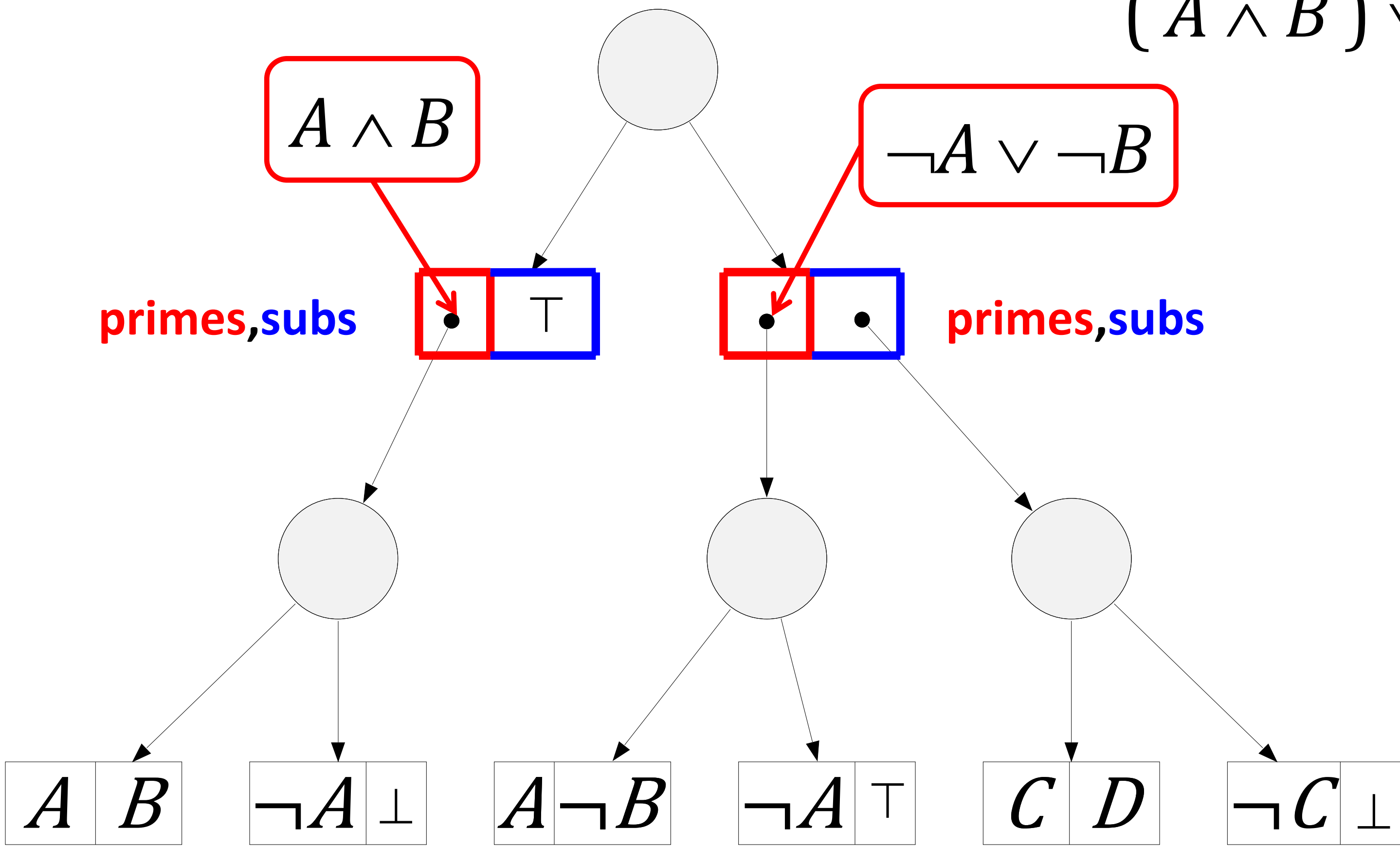
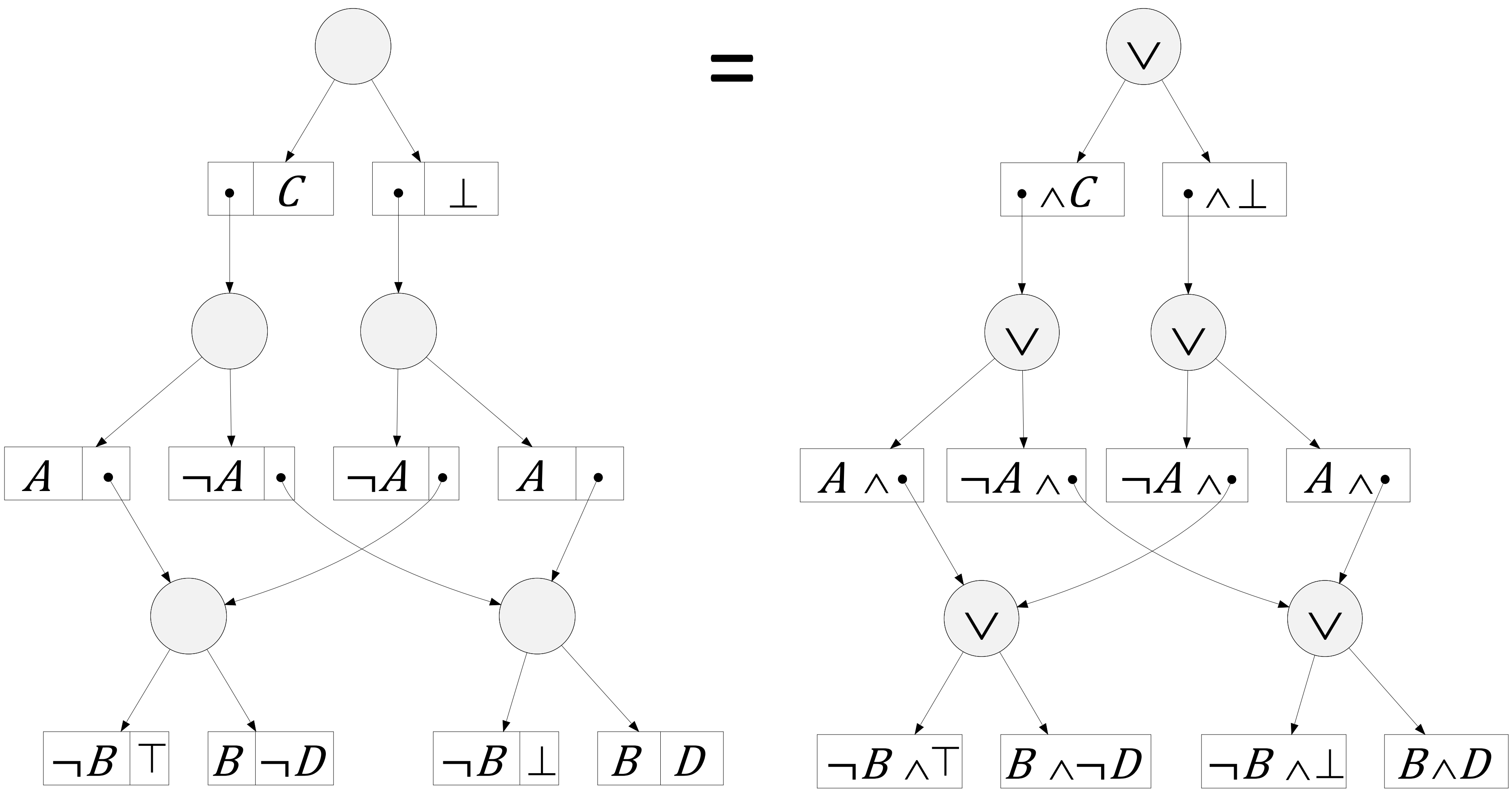


SDDs as Boolean Circuits

$$f(A, B, C, D) \ \ (A \oplus (B \wedge D)) \wedge C$$



(X, Y)-Partitions

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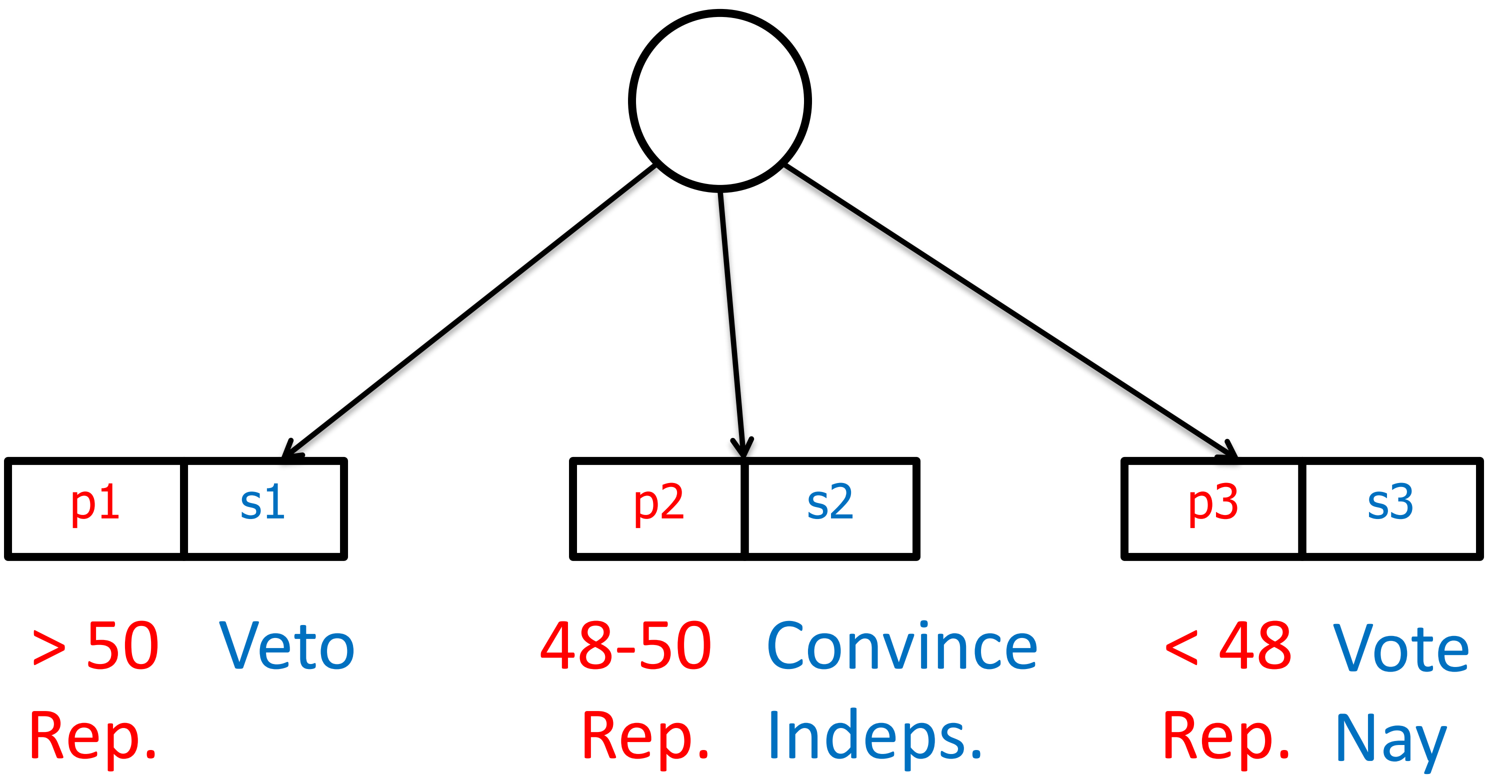

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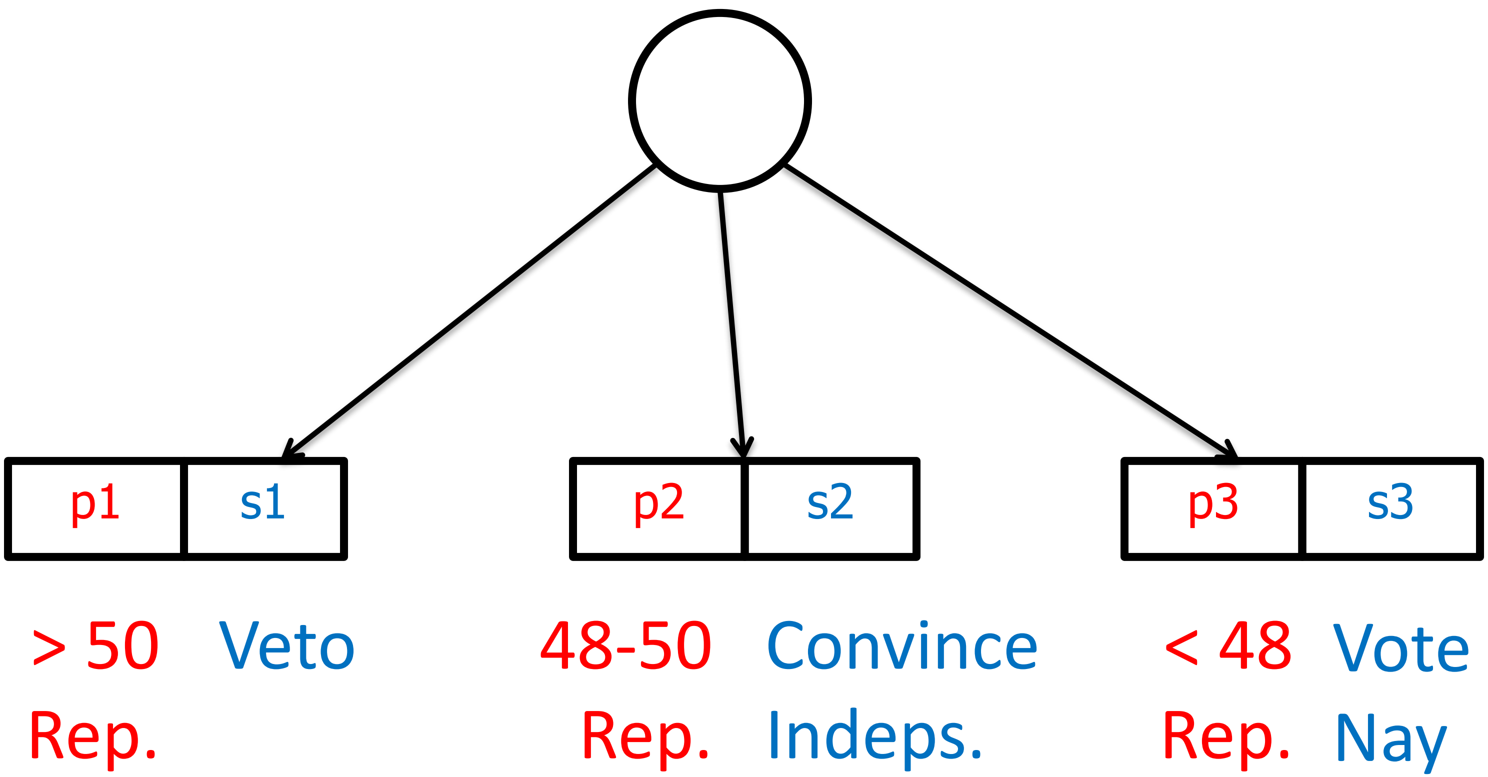

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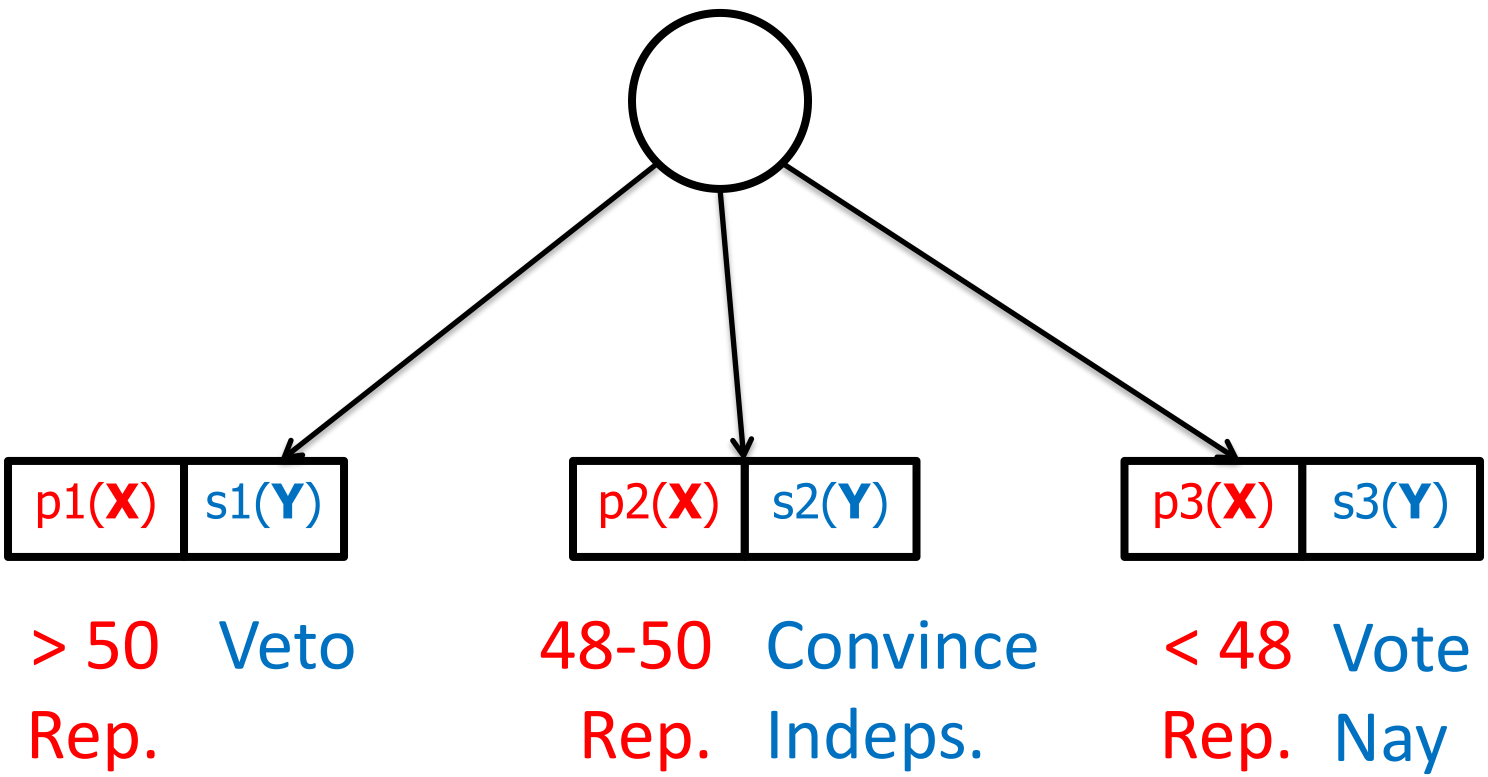

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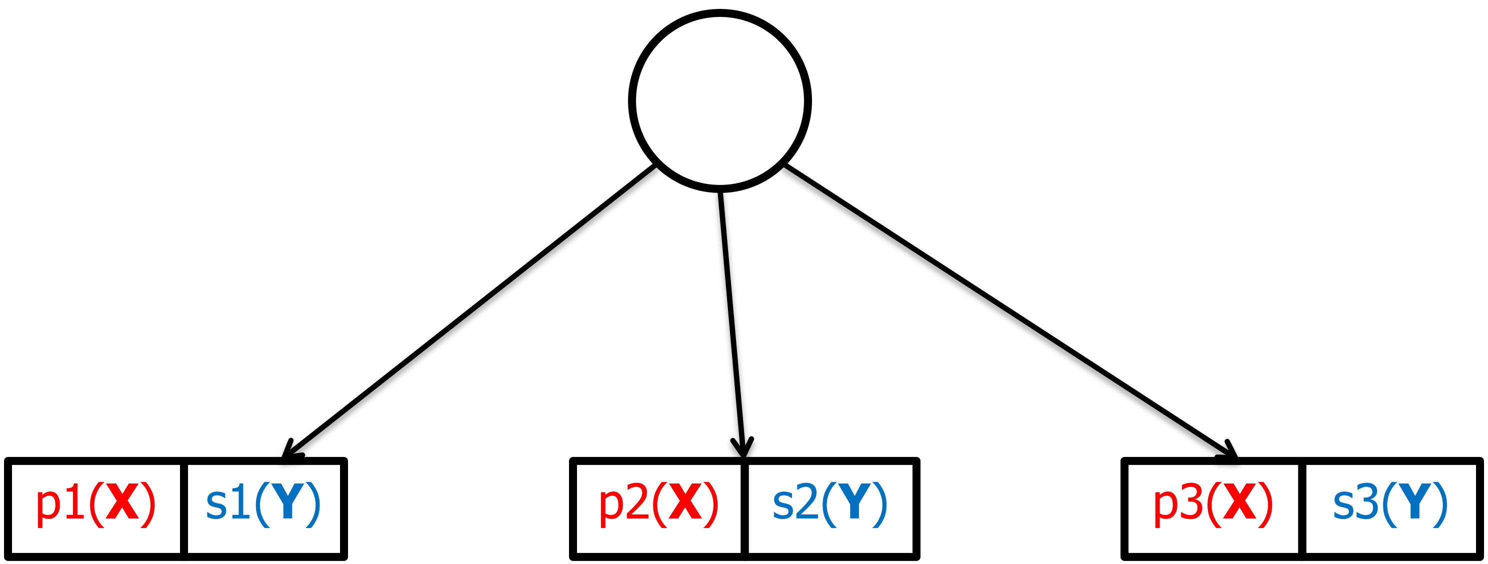
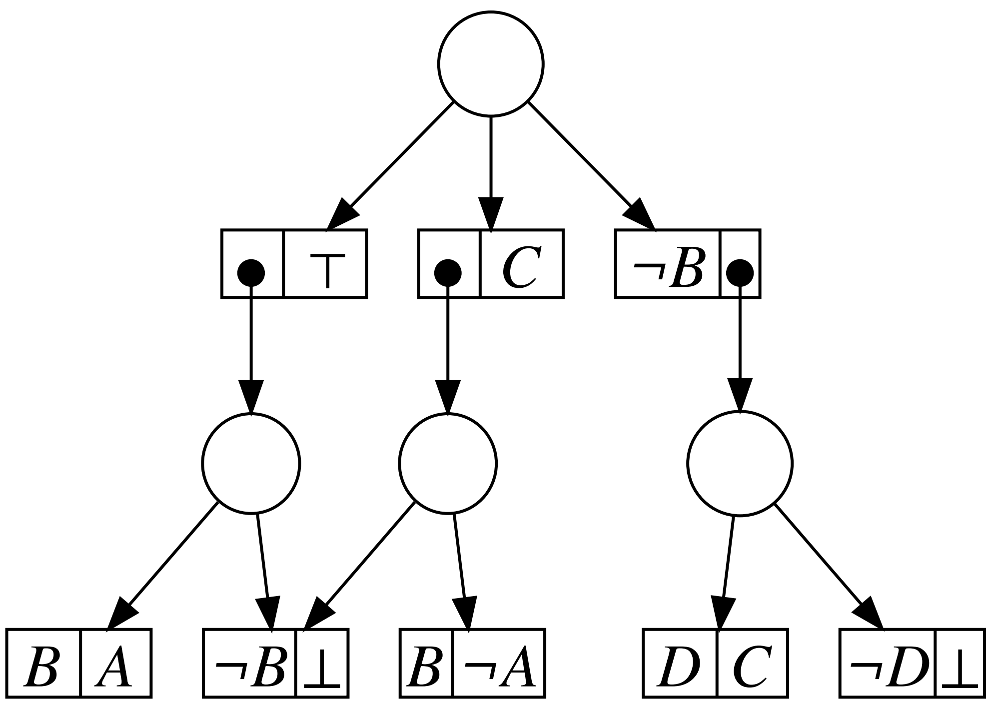


$$f(\mathbf{X}, \mathbf{Y}) = p_1(\mathbf{X})\, s_1(\mathbf{Y}) \vee \ldots \vee p_n(\mathbf{X})\, s_n(\mathbf{Y})$$

Variable **order** becomes variable **tree** (vtree)

$$f = (A \wedge B) \vee (B \wedge C) \vee (C \wedge D)$$



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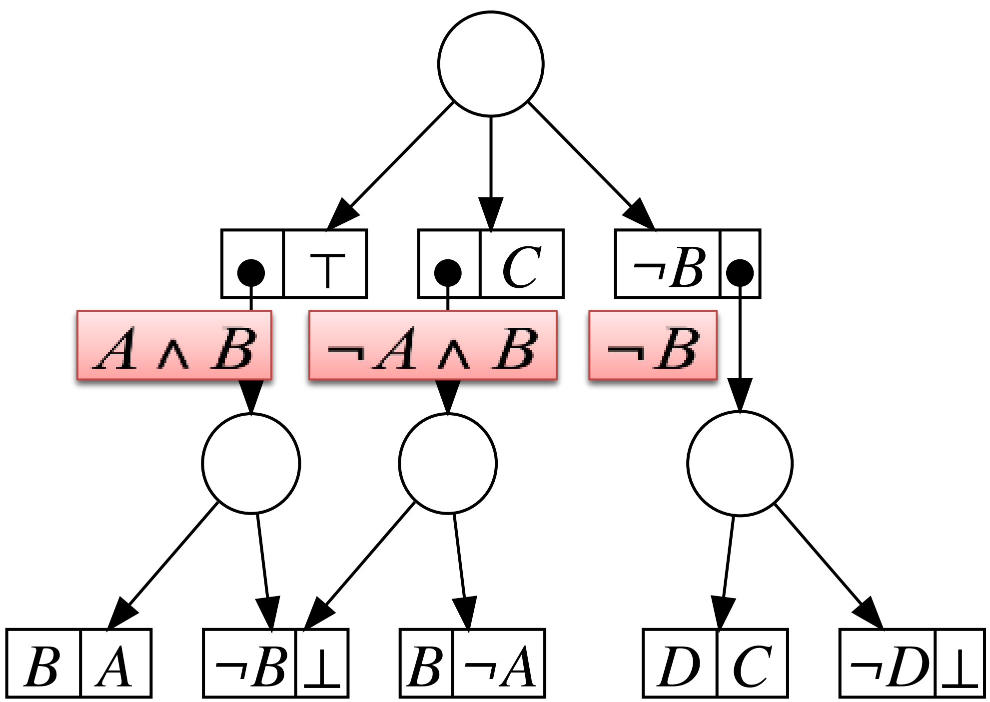

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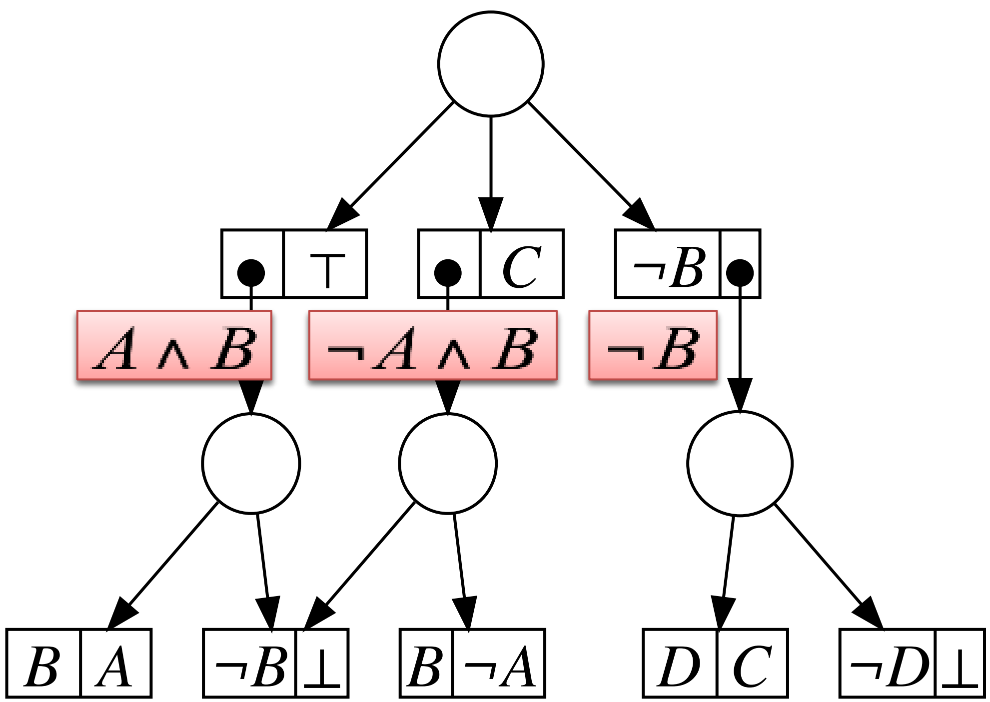

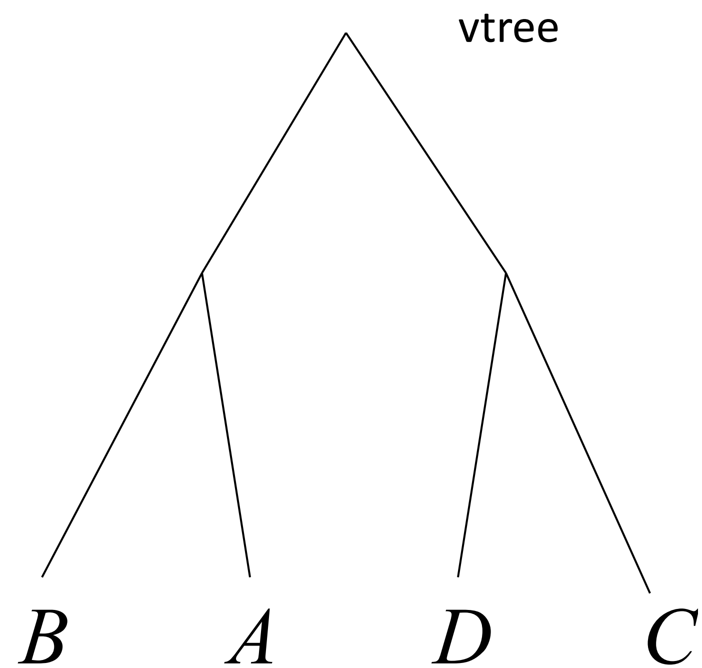

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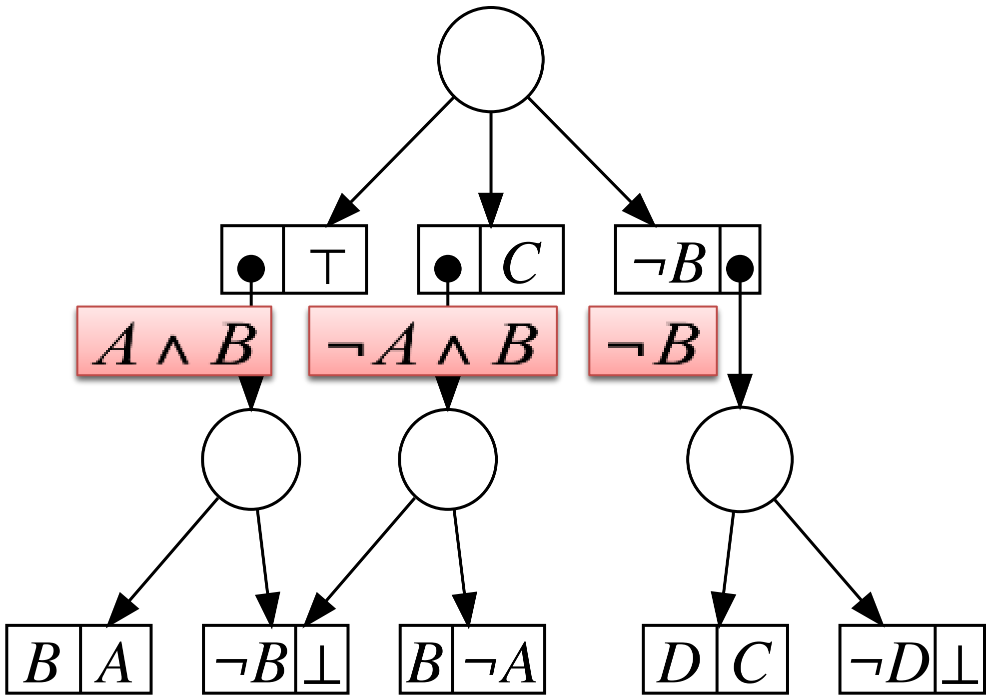

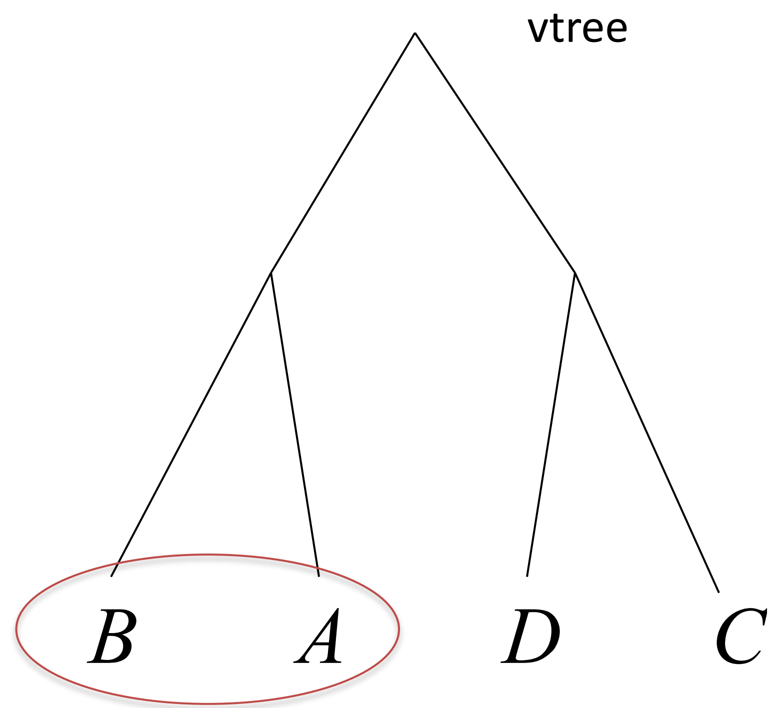

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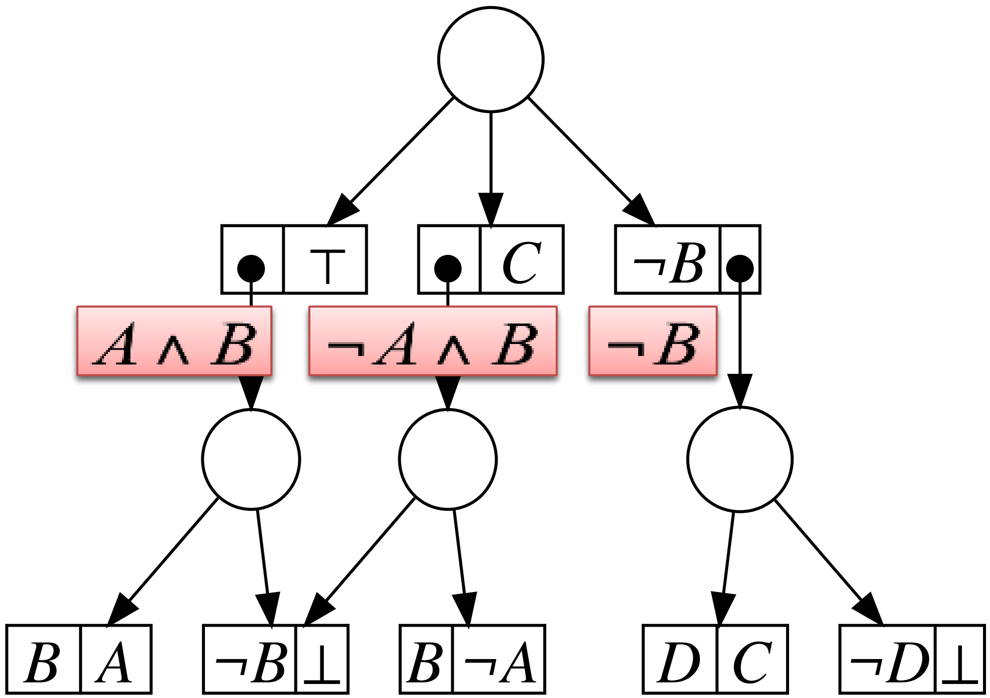
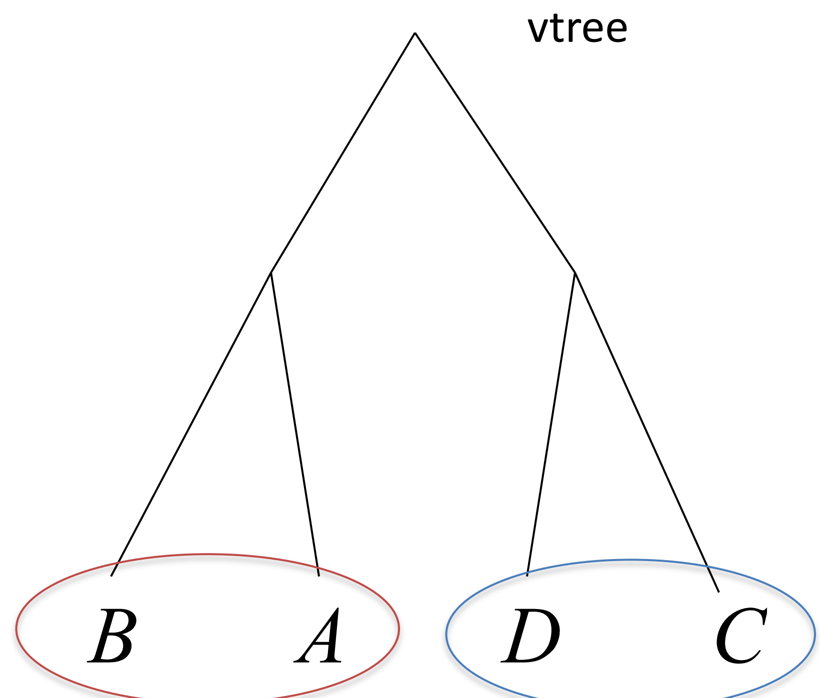
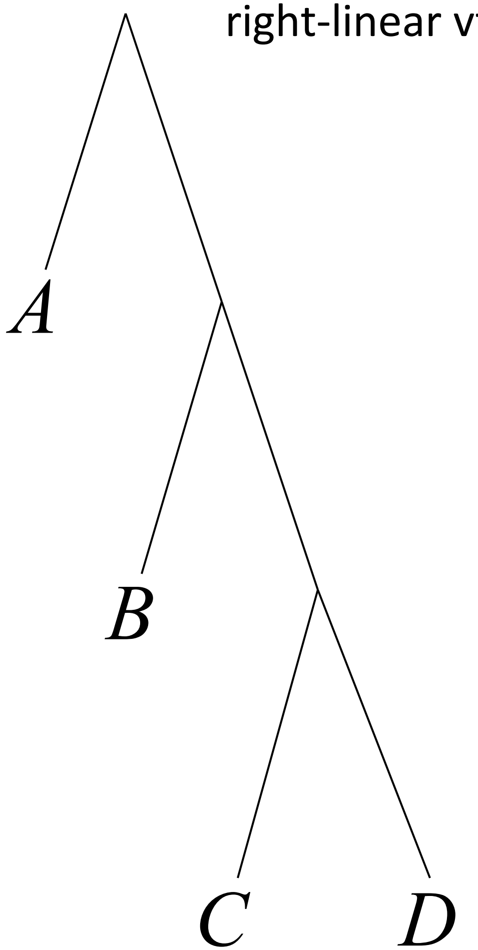




OBDDs are SDDs

right-linear vtree



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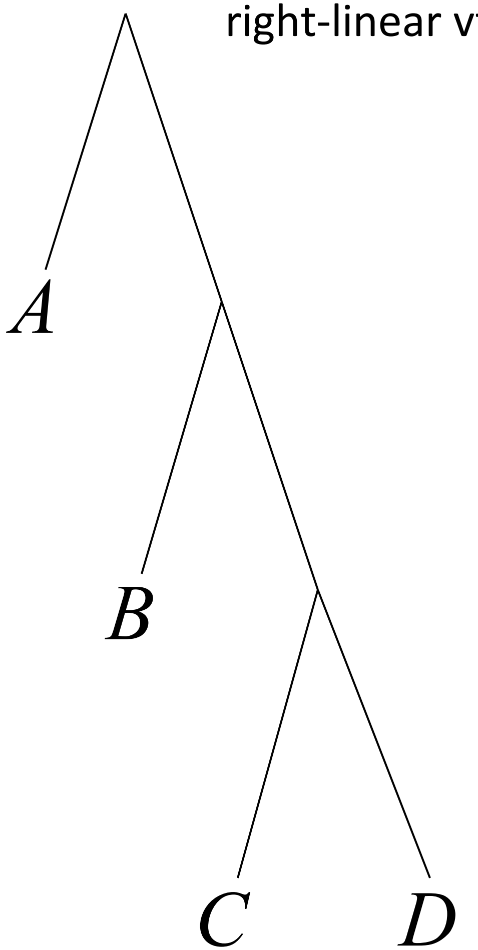

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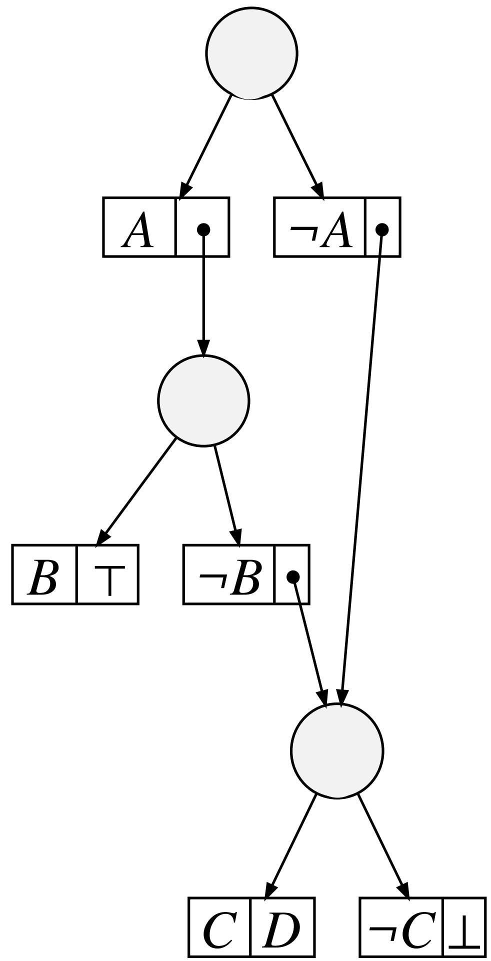

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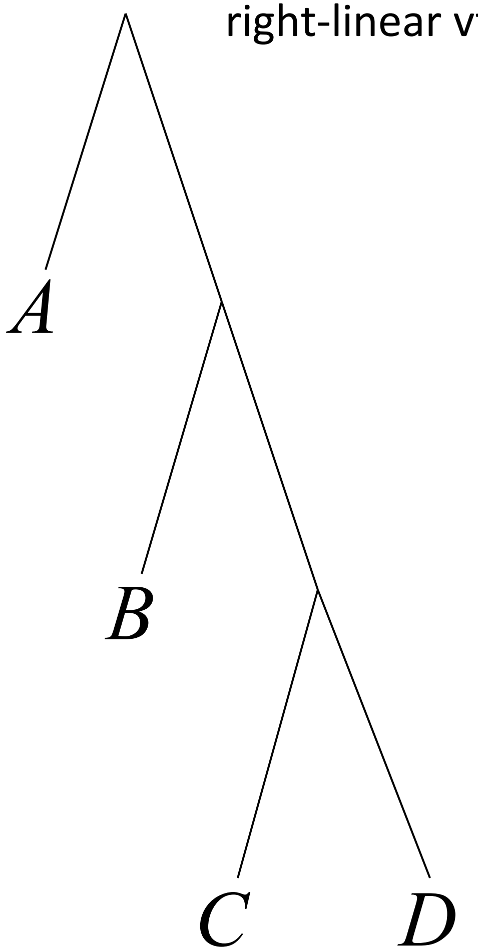

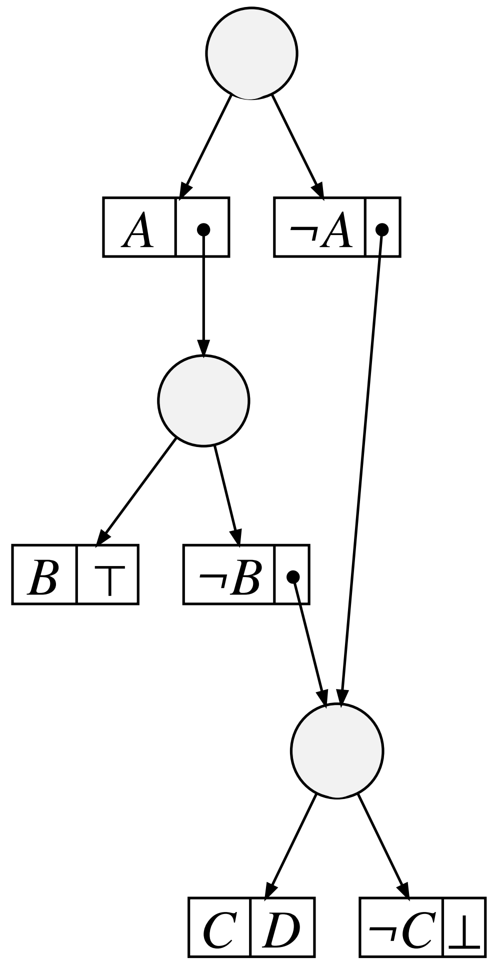

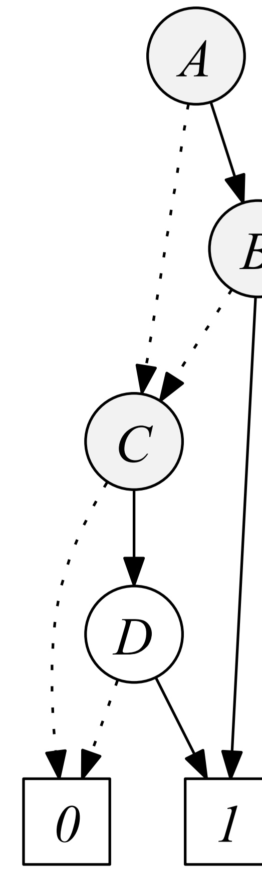

Ingredients for
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- M Minimization
- A Apply Function
- S Succinctness
- Q Queries

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- **Minimization**
- Apply Function
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- Queries

Compression

- An (**X**,**Y**)-partition: $f\,(\mathbf{X},\,\mathbf{Y}) = p_1(\mathbf{X})s_1(\mathbf{Y}) \vee \ldots \vee p_n(\mathbf{X})s_n(\mathbf{Y})$

  is ***compressed*** when subs are distinct: $s_\mathrm{i}(\mathbf{Y}) \neq s_\mathrm{i}(\mathbf{Y})$ if i≠j
- f(**X**,**Y**) has a **unique** compressed **(X,Y)**-partition

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prime	sub
$A \wedge B$	true
$A \wedge \overline{B}$	$C \wedge D$
$\overline{A} \wedge B$	C
$\overline{A} \wedge \overline{B}$	$C \wedge D$

Compression

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$\overline{A} \wedge B$	C
$\overline{A} \wedge \overline{B}$	$C \wedge D$

$\vee$

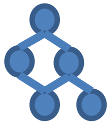
prime	sub
$A \wedge B$	true
$\overline{A} \wedge B$	C
$\overline{B}$	$C \wedge D$

SDDs are Canonical

For a fixed vtree (fixing **X**,**Y** throughout the SDD), compressed SDDs are **canonical**!

Equivalent sentences have identical circuits.

$$A \wedge (C \vee D) \quad \equiv \quad (A \wedge C) \vee (A \wedge D)$$



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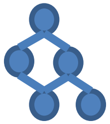


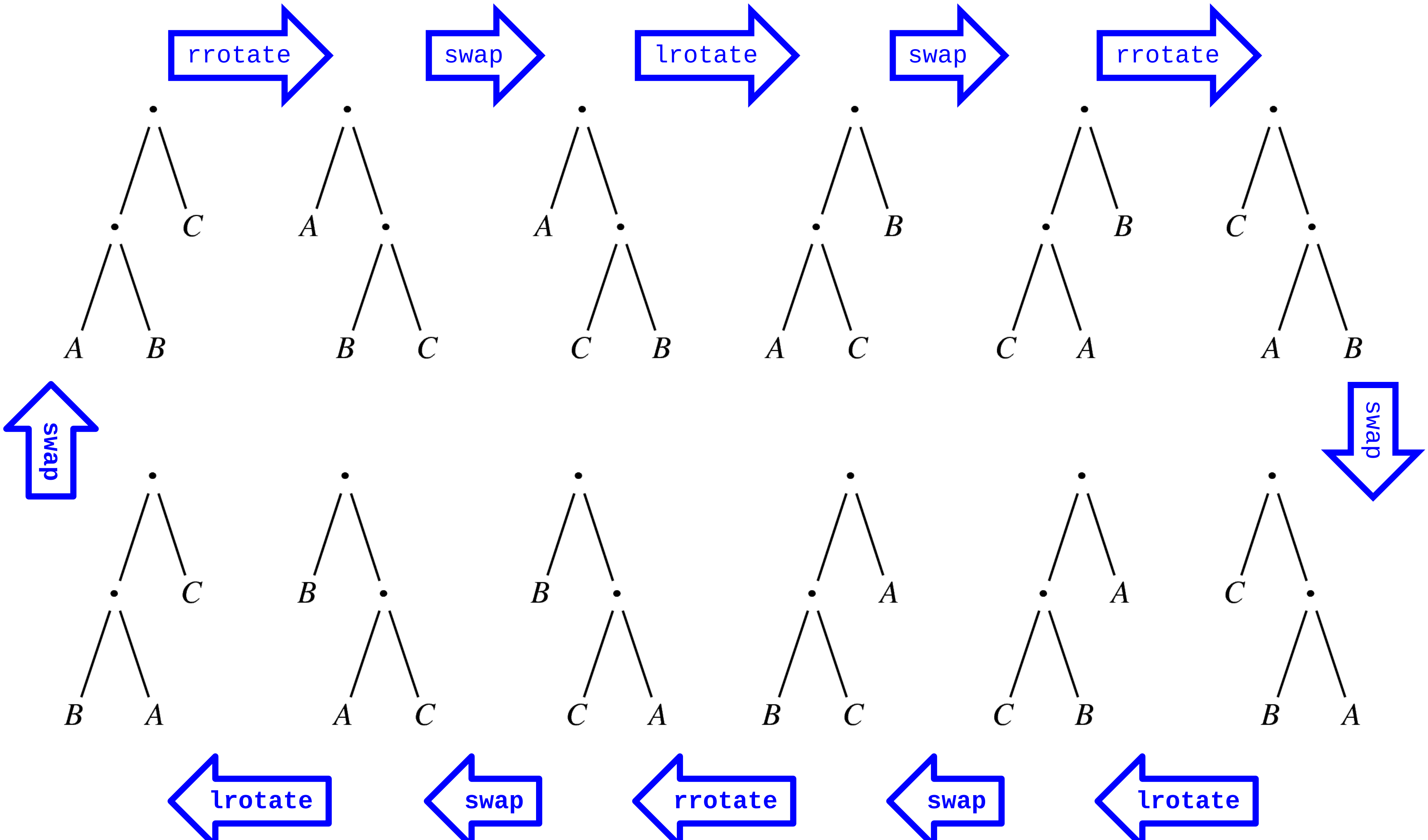
OBDD Minimization

- 24 **ordering** of 4 variables

$$ABCD \Rightarrow ABDC \Rightarrow ADBC \Rightarrow DABC \Rightarrow DACB \Rightarrow ADCB \Rightarrow$$
$$ACDB \Rightarrow ACBD \Rightarrow CABD \Rightarrow CADB \Rightarrow CDAB \Rightarrow DCAB \Rightarrow$$
$$DCBA \Rightarrow CDBA \Rightarrow CBDA \Rightarrow CBAD \Rightarrow BCAD \Rightarrow BCDA \Rightarrow$$
$$BDCA \Rightarrow DBCA \Rightarrow DBAC \Rightarrow BDAC \Rightarrow BADC \Rightarrow BACD$$

- 24 OBDDs for every function over 4 variables
- Searching for an optimal OBDD is searching for an **optimal variable order**

SDD Minimization



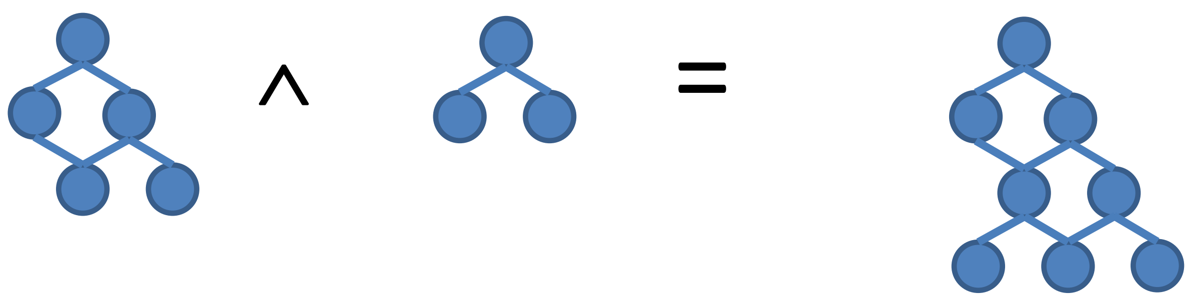
Ingredients for *Delicious Decision Diagrams*

- Minimization
- **Apply Function**
- Succinctness
- Queries

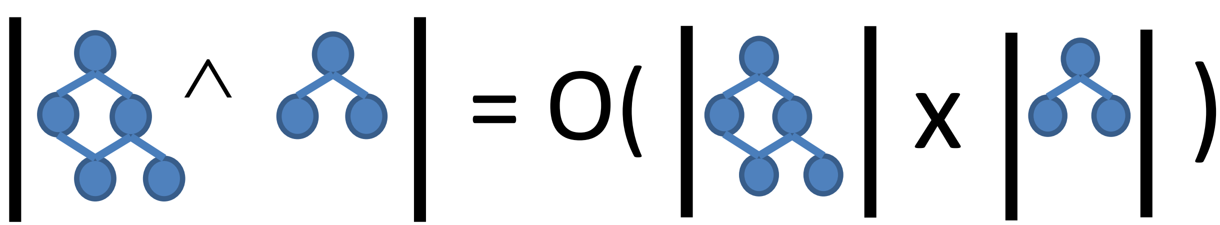
Efficient Apply Function

- Build Boolean **combinations** of existing circuits
- Compile **arbitrary** sentence incrementally

$(A \oplus (B \wedge D))$ $\qquad$ $(C \vee D)$ $\qquad$ $(A \oplus (B \wedge D)) \wedge (C \vee D)$



- **Polytime** Apply: one Apply cannot blow up size



Is Apply for SDDs Polytime?

Algorithm 1 $\mathtt{Apply}(\alpha, \beta, \circ)$

if α and β are constants or literals **then**
    return $\alpha \circ \beta$ *// result is a constant or literal*
else if $\mathtt{Cache}(\alpha, \beta, \circ) \neq \mathtt{nil}$ **then**
    return $\mathtt{Cache}(\alpha, \beta, \circ)$ *// has been computed before*
else
    $\gamma \leftarrow \{\}$
    for all elements (p_i, s_i) in α **do**
        for all elements (q_j, r_j) in β **do**
            $p \leftarrow \boxed{\mathtt{Apply}(p_i, q_j, \wedge)}$
            if p is consistent **then**
                $s \leftarrow \boxed{\mathtt{Apply}(s_i, r_j, \circ)}$
                add element (p, s) to γ

    return $\mathtt{Cache}(\alpha, \beta, \circ) \leftarrow \mathtt{UniqueD}(\gamma)$

- $|\alpha| \times |\beta|$ recursive calls
- **Polytime**!

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Succinctness

- Theory
  - OBDD $\subset$ SDD thus SDD **never larger** than OBDD
  - Quasi-polynomial separation with OBDD

    *OBDD can be **much larger** than SDD*
  - **Treewidth** upper bounds (important in AI!)
- Practice
  - SDD Compiler available and effective
  - SDD Package: http://reasoning.cs.ucla.edu/sdd/
  - Can obtain orders of magnitude improvements

Ingredients for *Delicious Decision Diagrams*

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- Apply Function
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- **Queries**

Queries

- OBDDs are Swiss army knife of supported queries
- SDDs are **equally powerful**

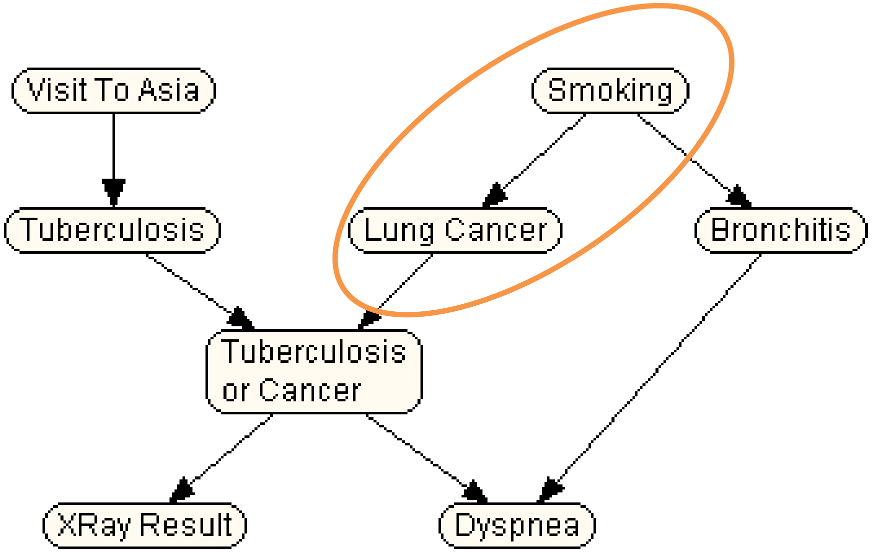
Query	Description	OBDD	SDD
CO	consistency	√	√
VA	validity	√	√
CE	clausal entailment	√	√
IM	implicant check	√	√
EQ	equivalence check	√	√
CT	model counting	√	√
SE	sentential entailment	√	√
ME	model enumeration	√	√

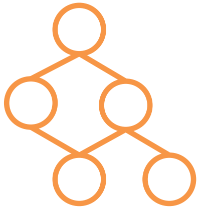
- Some enabled by canonicity + apply 

- E.g., (Weighted) Model Counting for Probabilistic reasoning (E.g., Pr(bill passes|Vote1=Yea))

Application: Bayesian Networks

- Incrementally compile network 





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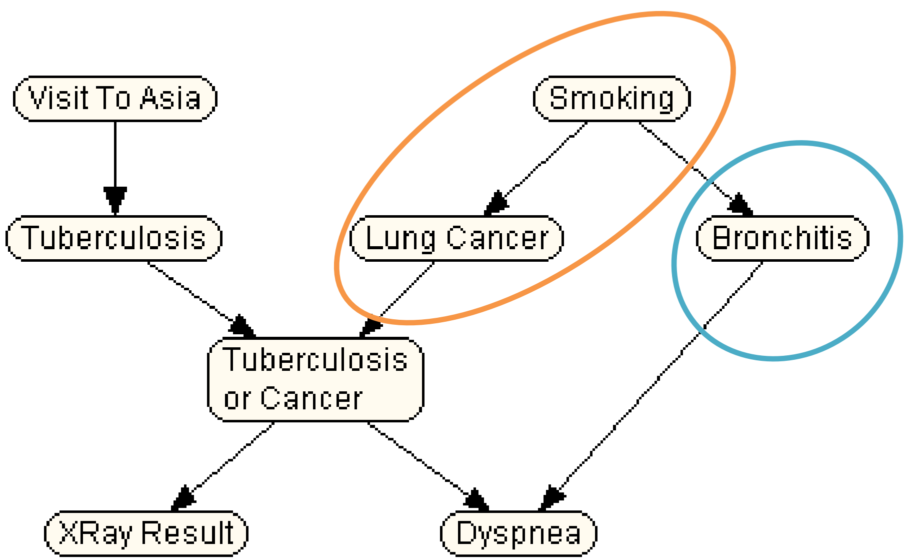

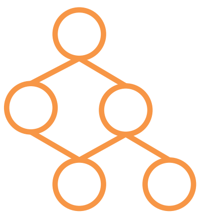

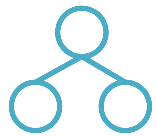

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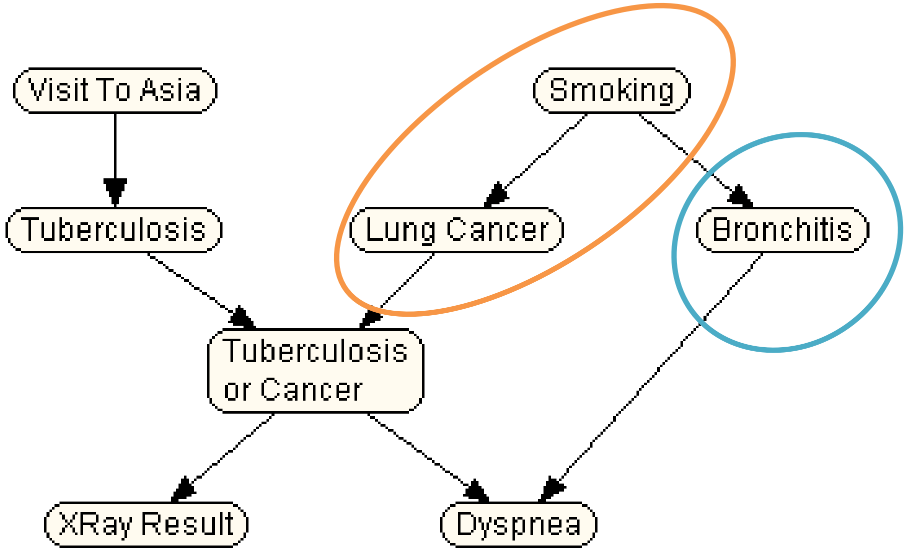

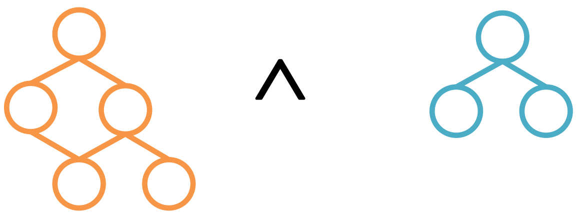

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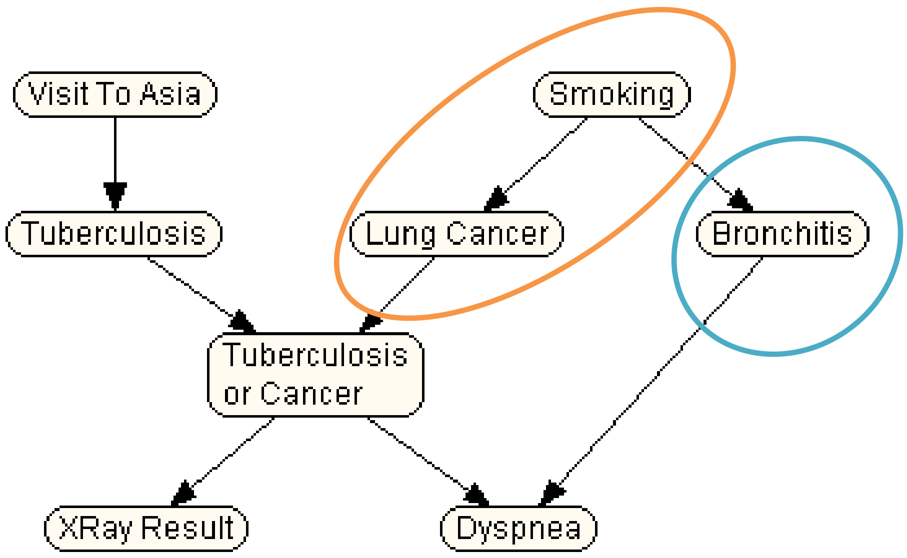

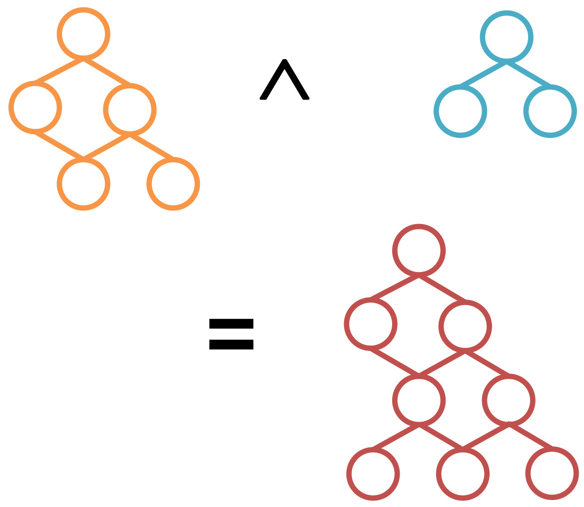

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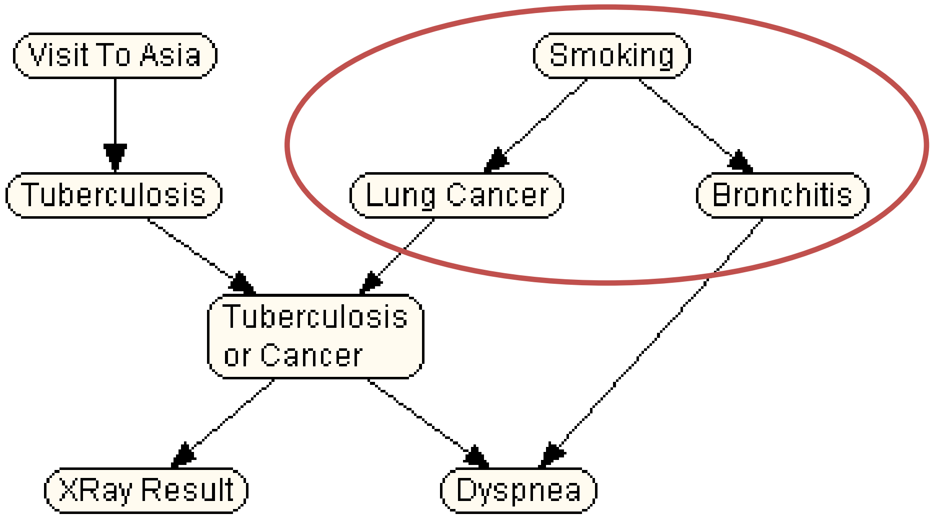
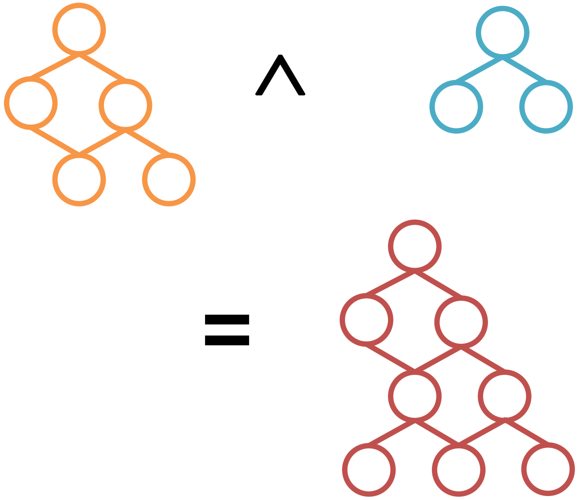
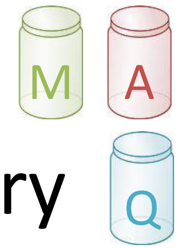
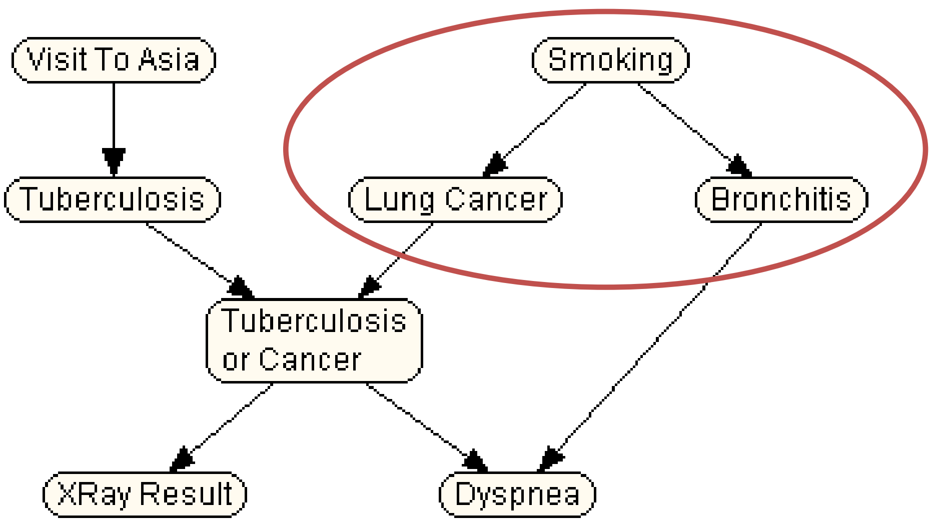


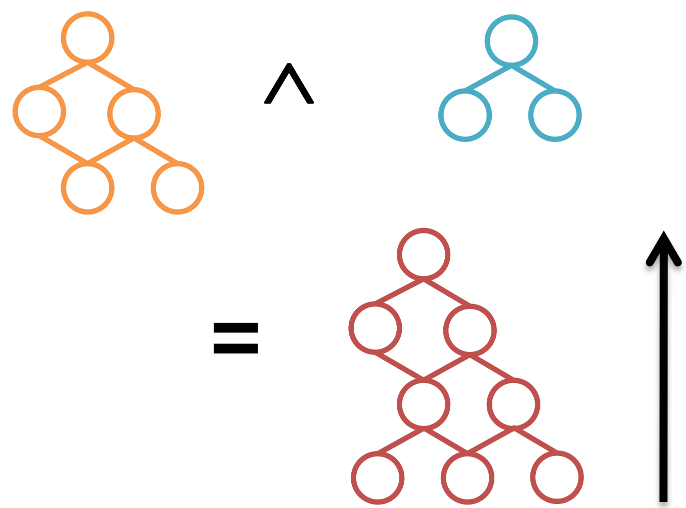


Application: Bayesian Networks

- Incrementally compile network
- Compute probability of any query





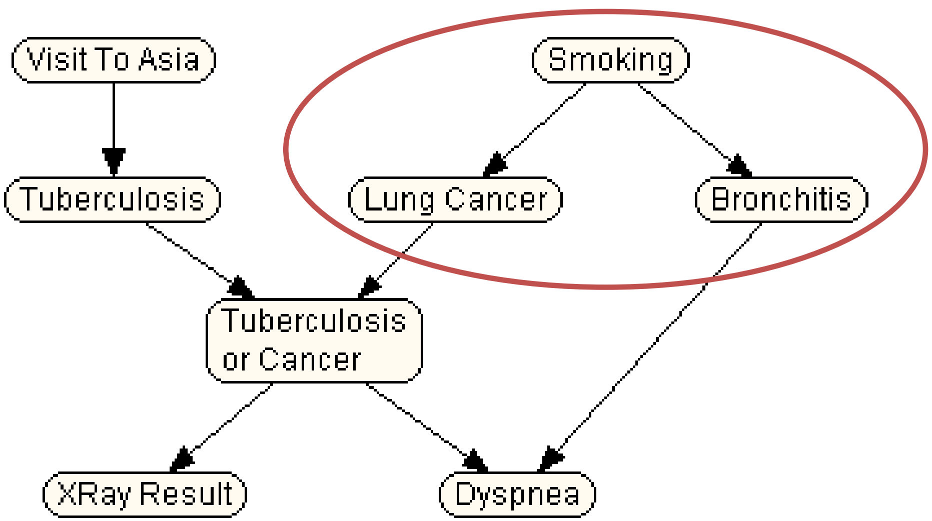


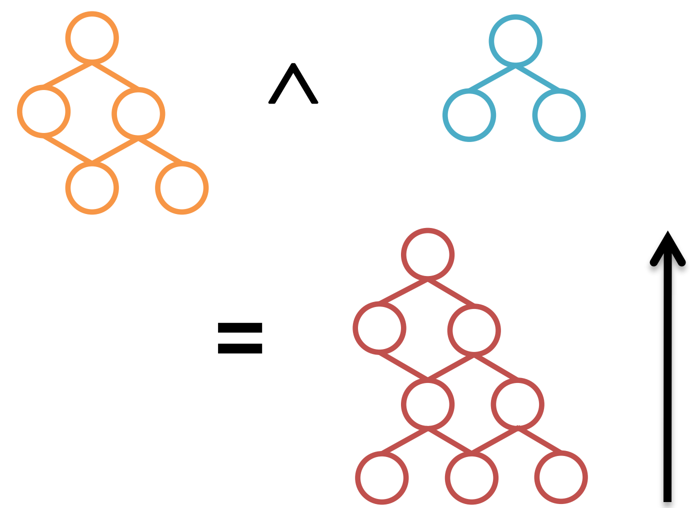
Application: Bayesian Networks

- Incrementally compile network 

- Compute probability of any query 

- Better than state of the art (treewidth) 

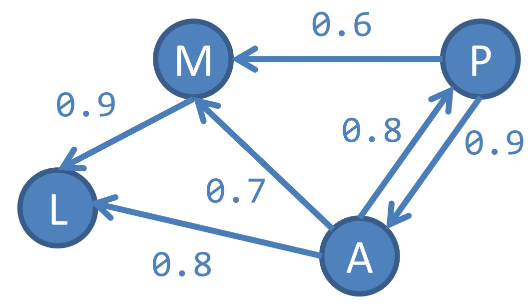




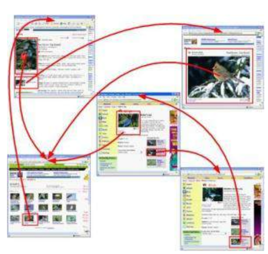
Application: Probabilistic Programming

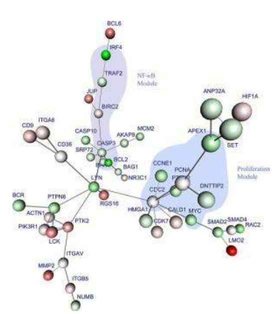
Model = program with random numbers

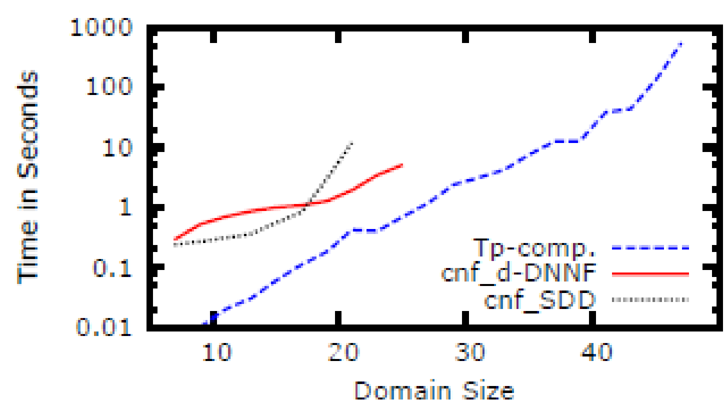
```
reach(X,Y) :- flight(X,Y).
reach(X,Y) :- flight(X,Z), reach(Z,Y).
```



State of the art inference: SDDs







Application: Tractable Learning

- Given: data
- Objective:
  - learn a probability distribution
  - ensure distribution is tractable for querying
- Unstructured space: Voting data
- Structured space: Movie recommendation

Learning in Unstructured Spaces

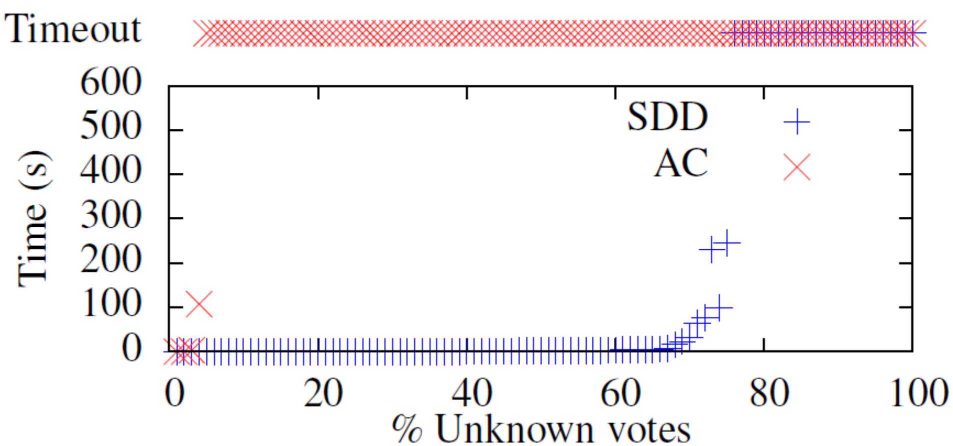
- Voting data from US House

  1764 votes of 453 congressmen

- Learn distribution (Markov network) 

- Represent as SDD to ensure tractability 

- Query efficiency 



Learning in Structured Spaces

Student enrollment constraints:

- Must take at least one of Probability or Logic.

$$P \vee L$$

- Probability is a prerequisite for AI.

$$A \Rightarrow P$$

- The prerequisites for KR is either AI or Logic.

$$K \Rightarrow (P \vee L)$$

$w = A \wedge K \wedge L \wedge \neg P$ ***impossible***

L	K	P	A	Students
0	0	1	0	6
0	0	1	1	54
0	1	1	1	10
1	0	0	0	5
1	0	1	0	1
1	0	1	1	0
1	1	0	0	17
1	1	1	0	4
1	1	1	1	3

Table 1: Student enrollment data.

Example: Rankings and Permutations

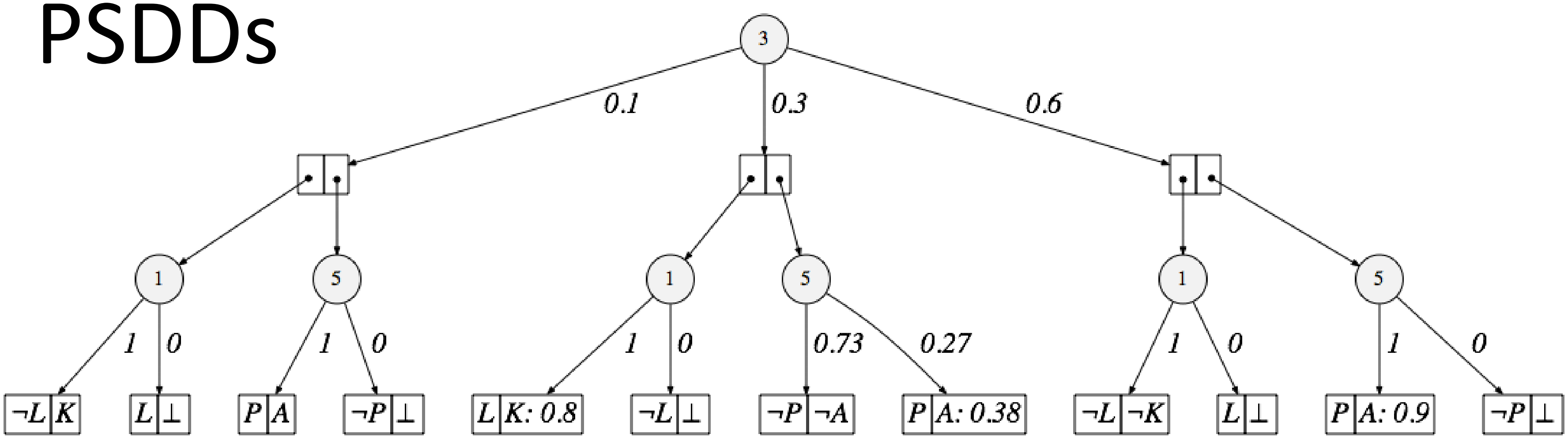
rank	user 1
1	The Godfather
2	Raiders of the Lost Ark
3	Casablanca
4	The Shawshank Redemption
5	Schindler's List
⋮	⋮

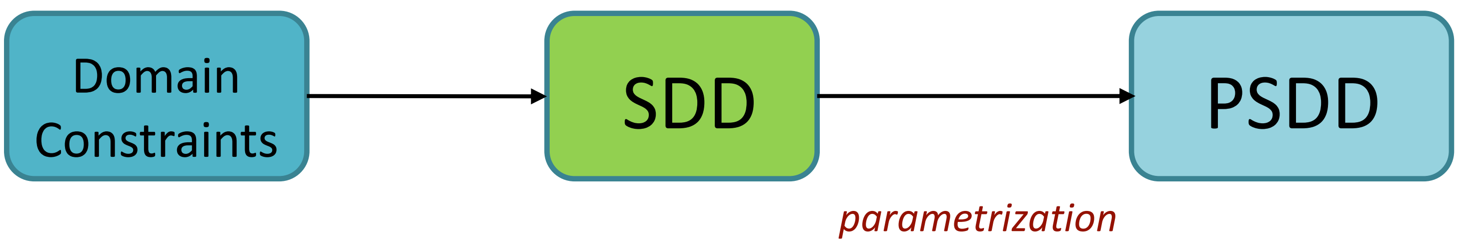
rank	user 2
1	Star Wars V: The Empire Strikes Back
2	Star Wars IV: A New Hope
3	The Godfather
4	The Shawshank Redemption
5	The Usual Suspects
⋮	⋮

rank	user 3
1	The Usual Suspects
2	One Flew over the Cuckoo's Nest
3	The Godfather: Part II
4	Monty Python and the Holy Grail
5	Star Wars IV: A New Hope
⋮	⋮

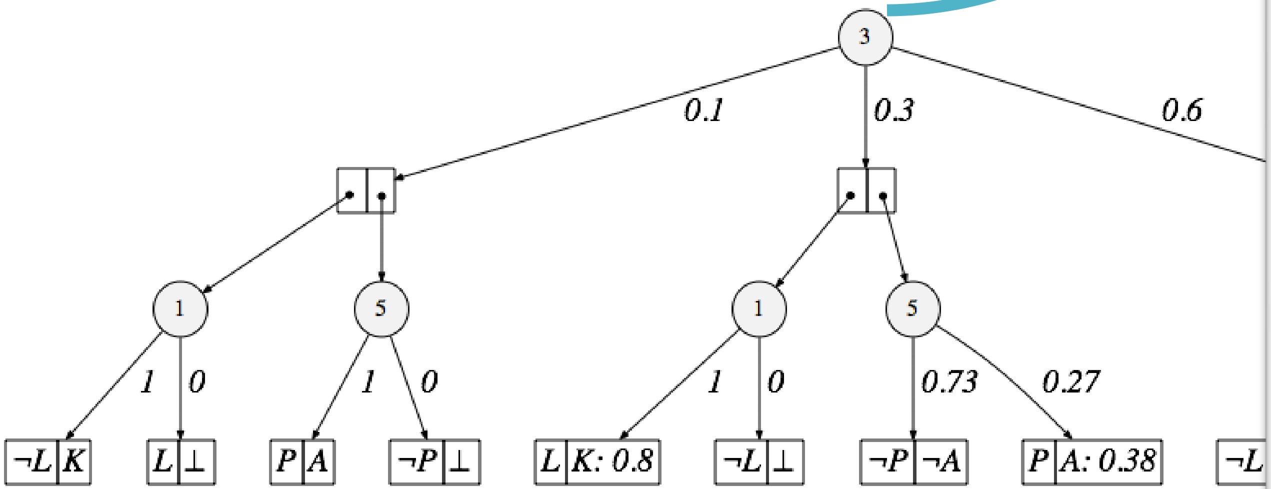
Learn rankings of movies (permutations):
Predict new movies given preferences

Distributions over Structured Spaces: PSDDs



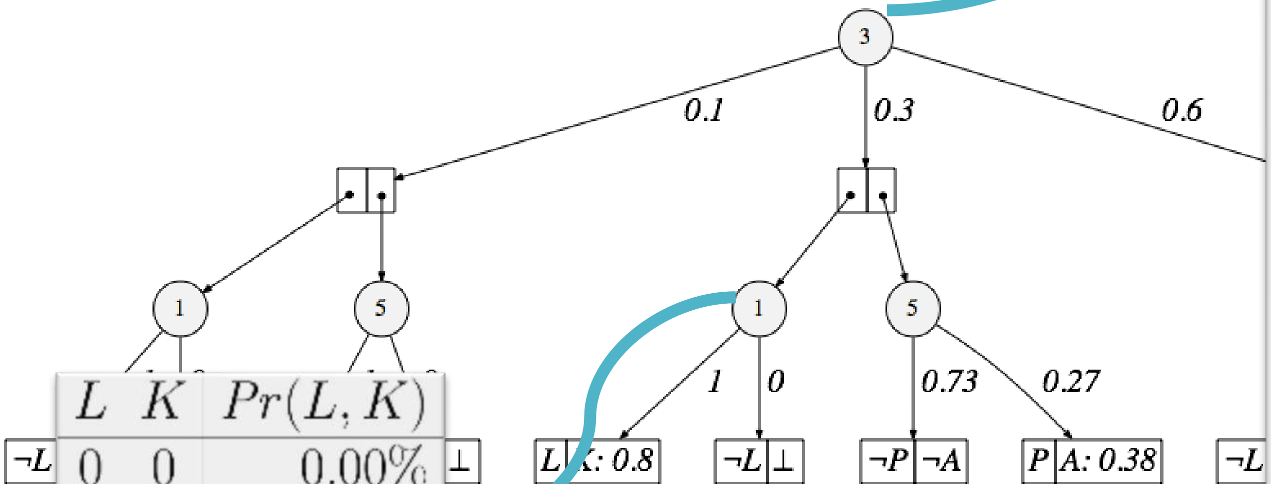


Distribution



L	K	P	A	$Pr(L,K,P,A)$
0	0	0	0	0.00%
0	0	0	1	0.00%
0	0	1	0	6.00%
0	0	1	1	54.00%
0	1	0	0	0.00%
0	1	0	1	0.00%
0	1	1	0	0.00%
0	1	1	1	10.00%
1	0	0	0	4.40%
1	0	0	1	0.00%
1	0	1	0	1.00%
1	0	1	1	0.60%
1	1	0	0	17.6%
1	1	0	1	0.00%
1	1	1	0	4.00%
1	1	1	1	2.40%

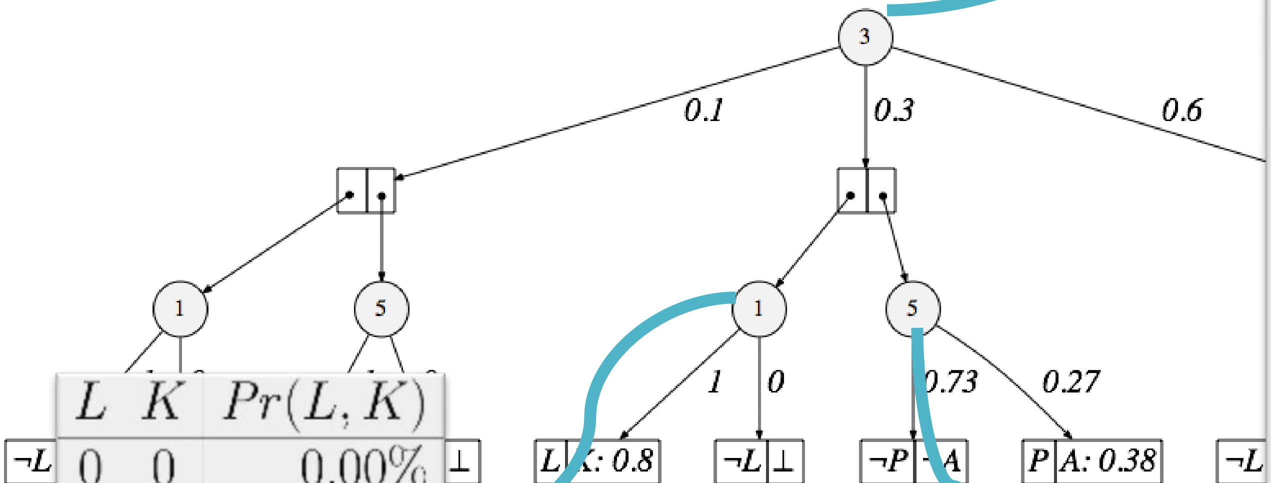
Distribution



L	K	$Pr(L,K)$
0	0	0.00%
0	1	0.00%
1	0	20.00%
1	1	80.00%

L	K	P	A	$Pr(L,K,P,A)$
0	0	0	0	0.00%
0	0	0	1	0.00%
0	0	1	0	6.00%
0	0	1	1	54.00%
0	1	0	0	0.00%
0	1	0	1	0.00%
0	1	1	0	0.00%
0	1	1	1	10.00%
1	0	0	0	4.40%
1	0	0	1	0.00%
1	0	1	0	1.00%
1	0	1	1	0.60%
1	1	0	0	17.6%
1	1	0	1	0.00%
1	1	1	0	4.00%
1	1	1	1	2.40%

Distribution



L	K	$Pr(L,K)$
0	0	0.00%
0	1	0.00%
1	0	20.00%
1	1	80.00%

P	A	$Pr(P,A)$
0	0	73.33%
0	1	0.00%
1	0	16.67%
1	1	10.00%

L	K	P	A	$Pr(L,K,P,A)$
0	0	0	0	0.00%
0	0	0	1	0.00%
0	0	1	0	6.00%
0	0	1	1	54.00%
0	1	0	0	0.00%
0	1	0	1	0.00%
0	1	1	0	0.00%
0	1	1	1	10.00%
1	0	0	0	4.40%
1	0	0	1	0.00%
1	0	1	0	1.00%
1	0	1	1	0.60%
1	1	0	0	17.6%
1	1	0	1	0.00%
1	1	1	0	4.00%
1	1	1	1	2.40%

Reasoning with PSDDs
Example: Preference Distributions

observe:
- favorite movie is Star Wars V

rank	movie
1	Star Wars V: The Empire Strikes Back
2	Star Wars IV: A New Hope
3	The Godfather
4	The Shawshank Redemption
5	The Usual Suspects

observe:
- favorite movie is Star Wars V
- no other Star Wars movie in top-5
- at least one comedy in top-5

rank	movie
1	Star Wars V: The Empire Strikes Back
2	American Beauty
3	The Godfather
4	The Usual Suspects
5	The Shawshank Redemption

Conclusions

- SDD a strict **superset** of OBDD:
  - Characterized by trees, which include orders
  - Branch over sentences, which include literals
- SDDs maintain key **properties** of OBDDs:
  - Canonical, Polytime* Apply, Queries, etc.

- SDDs are more **succinct**
  - Treewidth instead of pathwidth
- Lots of **applications** in probabilistic AI and ML

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